# Chapter 4

# Controllability

This chapter develops the fundamental results about controllability and pole assignment.

## 4.1 Reachable States

We study the linear system

$$\dot{x} = Ax + Bu, \quad t \geq 0,$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Thus $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We begin our study of controllability with a question: What states can we get to from the origin by choice of input? Are there states in $\mathbb{R}^n$ that are unreachable? This is a question of control authority. For example, if $B$ is the zero matrix, there is no control authority, and if we start $x$ at the origin, we'll stay there. By contrast, if $B = I$ it will turn out that we can reach every state.

Fix a time $t_1 > 0$. We say a vector $v$ in $\mathbb{R}^n$ is **reachable** (at time $t_1$) if there exists an input $u(\cdot)$ that steers the state from the origin at $t = 0$ to $v$ at $t = t_1$.

To characterize reachability we have to recall the integral form of the above differential equation. It was derived in ECE356 that

$$x(t) = \mathrm{e}^{tA}x(0) + \int_0^t \mathrm{e}^{(t-\tau)A}Bu(\tau)d\tau.$$

If $x(0) = 0$ and $t = t_1$, then

$$x(t_1) = \int_0^{t_1} \mathrm{e}^{(t_1-\tau)A}Bu(\tau)d\tau.$$

Thus a vector $v$ is reachable at time $t_1$ iff

$$(\exists u(\cdot))\ v = \int_0^{t_1} \mathrm{e}^{(t_1-\tau)A}Bu(\tau)d\tau.$$

Let us introduce the space $\mathcal{U}$ of all signals $u(\cdot)$ defined on the time interval $[0, t_1]$. The smoothness of $u(\cdot)$ is not really important, but to be specific, let's assume $u(\cdot)$ is continuous. Also, let us introduce the **reachability operator**

$$\mathbf{R} : \mathcal{U} \to \mathbb{R}^n,\ \mathbf{R}u = \int_0^{t_1} \mathrm{e}^{(t_1-\tau)A}Bu(\tau)\,d\tau.$$

In words, **R** is the linear transformation (LT) that maps the input signal to the state at time $t_1$ starting from $x(0) = 0$.

The LT **R** is not the same as a matrix because $\mathcal{U}$ isn't finite dimensional. But its image is well defined and Im **R** is a subspace of $\mathbb{R}^n$, as is easy to prove.

It's time now to introduce the **controllability matrix**

$$W_c = \left[ \begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array} \right].$$

In the single-input case, $B$ is $n \times 1$ and $W_c$ is square, $n \times n$. The importance of this matrix comes from the theorem to follow.

**Note** The object $B$ is a matrix. However, associated with it is an LT, namely the LT that maps a vector $u$ to the vector $Bu$. Instead of introducing more notation, we shall write Im $B$ for the image of this LT. That is, the symbol $B$ will stand for the matrix or the LT depending on the context. Likewise for other matrices such as $A$ and $W_c$.

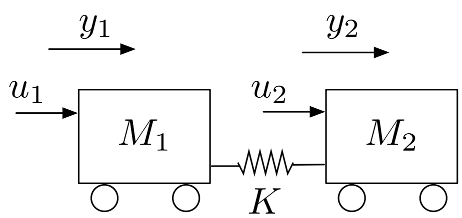
**Theorem 4.1.1** Im **R** = Im $W_c$, *i.e., the subspace of reachable states equals the column span of* $W_c$.

Let's postpone the proof and instead note the conclusion: A vector $v$ is reachable at time $t_1$ iff it belongs to the column span of $W_c$, i.e.,

$$\text{rank } W_c = \text{rank} \left[ \begin{array}{cc} W_c & v \end{array} \right].$$

Notice that reachability turns out to be independent of $t_1$. Also, every vector is reachable iff rank $W_c = n$.

**Example** Consider this setup of 2 carts, 2 forces:




The equations are

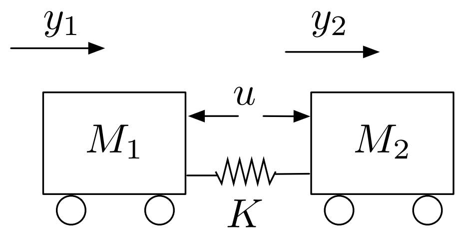
$$M_1\ddot{y}_1 = u_1 + K(y_2 - y_1), \quad M_2\ddot{y}_2 = u_2 + K(y_1 - y_2).$$

Taking the state $x = (y_1, \dot{y}_1, y_2, \dot{y}_2)$ and $M_1 = 1, M_2 = 1/2, K = 1$ we have the state model

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

We compute using Scilab/MATLAB (or by hand) that $W_c$ is $4 \times 8$ and its rank equals 4. Thus every state is reachable from the origin. That is, every position and velocity of the carts can be produced at any time by an appropriate open-loop control. In this sense, two forces gives enough control authority. □

**Example** Now consider 2 carts, 1 common force:




Now $u_1 = -u$ and $u_2 = u$. So $A$ is as before, while

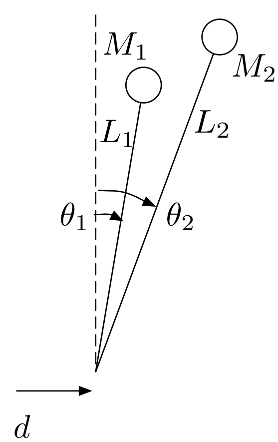
$$B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

Then

$$W_c = \begin{bmatrix} 0 & -1 & 0 & 3 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 0 & -6 \\ 2 & 0 & -6 & 0 \end{bmatrix}.$$

The rank equals 2. The set of reachable states is the 2-dimensional subspace spanned by the first two columns. So we don't have complete control authority in this case. □

**Example** Two pendula balanced on one hand:




The linearized equations of motion are

$$\begin{aligned} M_1(\ddot{d} + L_1\ddot{\theta}_1) &= M_1 g\theta_1 \\ M_2(\ddot{d} + L_2\ddot{\theta}_2) &= M_2 g\theta_2. \end{aligned}$$

Take the state and input to be

$$x = (\theta_1, \dot{\theta}_1, \theta_2\, \dot{\theta}_2), \quad u = \ddot{d}.$$

Find $W_c$ and show that every state is reachable iff $L_1 \neq L_2$. □

The proof of the theorem requires the Cayley-Hamilton theorem, which we discuss now. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Its characteristic polynomial is

$$s^2 + s + 1.$$

Substitute $s = A$ into this polynomial, regarding the constant as $s^0$. You get the matrix

$$A^2 + A + I.$$

Verify that this equals the zero matrix:

$$A^2 + A + I = 0.$$

Thus $A^2$ is a linear combination of $\{I, A\}$. Likewise for higher powers $A^3, A^4$ etc.

**Theorem 4.1.2** *If $p(s)$ denotes the characteristic polynomial of $A$, then $p(A) = 0$. Thus $A^n$ is a linear combination of lower powers of $A$.*

**Proof** Here's a proof for $n = 3$; the proof carries over for higher $n$. We have the identity

$$(sI - A)^{-1} = \frac{1}{p(s)} N(s),$$

where $p(s) = \det(sI - A)$ and $N(s)$ is the adjoint of $sI - A$. We can manipulate this to read

$$p(s)I = (sI - A)N(s).$$

Say $p(s) = s^3 + a_3 s^2 + a_2 s + a_1$. Then $N(s)$ must have the form $s^2 I + sN_2 + N_1$, where $N_1, N_2$ are constant matrices. Thus we have

$$(s^3 + a_3 s^2 + a_2 s + a_1)I = (sI - A)(s^2 I + sN_2 + N_1).$$

Equating coefficients of powers of $s$, we get

$$\begin{aligned} a_3 I &= N_2 - A \\ a_2 I &= N_1 - AN_2 \\ a_1 I &= -AN_1. \end{aligned}$$

Multiply the first equation by $A^2$ and the second by $A$, and then add all three: You get

$$a_3 A^2 + a_2 A + a_1 I = -A^3,$$

or

$$A^3 + a_3A^2 + a_2A + a_1I = 0.$$

□

**Proof of Theorem 4.1.1** We first show Im $\mathbf{R} \subset$ Im $W_c$. Let $v \in$ Im $\mathbf{R}$. Then $\exists u \in \mathcal{U}$ such that $v = \mathbf{R}u$, i.e.,

$$\begin{aligned} v &= \int_0^{t_1} e^{(t_1-\tau)A}Bu(\tau)\, d\tau \\ &= \int_0^{t_1} \left[ I + (t_1-\tau)A + \frac{(t_1-\tau)^2}{2!}A^2 + \cdots \right] Bu(\tau)\, d\tau \\ &= B\int_0^{t_1} u(\tau)d\tau + AB\int_0^{t_1} (t_1-\tau)u(\tau)d\tau + \cdots . \end{aligned}$$

Thus $v$ belongs to the column span of

$$\left[ \begin{array}{cccc} B & AB & A^2B & \cdots \end{array} \right].$$

By the Cayley-Hamilton theorem, the column span terminates at

$$\left[ \begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array} \right].$$

Now we show Im $W_c \subset$ Im $\mathbf{R}$. An equivalent condition is in terms of orthogonal complements: (Im $\mathbf{R})^\perp \subset$ (Im $W_c)^\perp$. So let $v$ be a vector orthogonal to Im $\mathbf{R}$. Thus for every $u(\cdot)$

$$v^T \int_0^{t_1} e^{(t_1-\tau)A}Bu(\tau)d\tau = 0.$$

That is,

$$\int_0^{t_1} v^T e^{(t_1-\tau)A}Bu(\tau)d\tau = 0.$$

Since this is true for every $u(\cdot)$, it must be that

$$v^T e^{(t_1-\tau)A}B = 0 \;\; \forall \tau, \; 0 \le \tau \le t_1.$$

This implies that

$$v^T e^{tA}B = 0 \;\; \forall t, \; 0 \le t \le t_1$$

and hence that

$$v^T \left( I + tA + \frac{t^2}{2}A^2 + \cdots \right) B = 0 \;\; \forall t, \; 0 \le t \le t_1.$$

Thus

$$v^T B = 0, \; v^T AB = 0, \ldots .$$

That is, $v$ is orthogonal to every column of $W_c$. □

An auxiliary question is, if $v$ is a reachable state, what control input steers the state from $x(0) = 0$ to $x(t_1) = v$? That is, if $v \in \text{Im}\,\mathbf{R}$, solve $v = \mathbf{R}u$ for $u$. There are an infinite number of $u$ because the nullspace of $\mathbf{R}$ is nonzero. We can get one input as follows. We want to solve

$$v = \int_0^{t_1} \mathrm{e}^{(t_1-\tau)A} B u(\tau) d\tau$$

for the function $u$. Without any motivation, let's look for a solution of the form

$$u(\tau) = B^T \mathrm{e}^{(t_1-\tau)A^T} w,$$

where $w$ is a constant vector. Then the equation to be solved is

$$v = \int_0^{t_1} \mathrm{e}^{(t_1-\tau)A} B B^T \mathrm{e}^{(t_1-\tau)A^T} w d\tau.$$

Now $w$ can be brought outside the integral:

$$v = \left[ \int_0^{t_1} \mathrm{e}^{(t_1-\tau)A} B B^T \mathrm{e}^{(t_1-\tau)A^T} d\tau \right] w.$$

In square brackets is a square matrix:

$$L_c = \int_0^{t_1} \mathrm{e}^{(t_1-\tau)A} B B^T \mathrm{e}^{(t_1-\tau)A^T} d\tau.$$

If $L_c$ is invertible, we're done, because $w = L_c^{-1} v$. It can be shown that $L_c$ is in fact invertible if $W_c$ has full rank.

Recap: The set of all states reachable starting from the origin is a subspace, the image (column span) of $W_c$, the controllability matrix. Thus, if this matrix has rank $n$, every state is reachable. If a state is reachable at some time, then it's reachable at any time. Of course you'll have to use a big control if the time is very short.

Finally, we say that the pair of matrices $(A, B)$ is a **controllable pair** if every state is reachable, equivalently, the rank of $W_c$ equals $n$. Go over the examples in this section and see which are controllable.

## 4.2 Properties of Controllability

### Invariance under change of basis

The state vector of a system is certainly not unique. For example, if $x$ is a state vector, so is $Vx$ for any square invertible matrix $V$. Suppose we have the state model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

and we define a new state vector, $\tilde{x} = Vx$. The new equations are

$$\begin{aligned} \dot{\tilde{x}} &= VAV^{-1}\tilde{x} + VBu \\ y &= CV^{-1}\tilde{x} + Du. \end{aligned}$$

Under the change of state $x \longmapsto Vx$, the $A, B$ matrices change like this

$$(A, B) \longmapsto (VAV^{-1}, VB)$$

and the controllability matrix changes like this

$$\begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} \longmapsto V \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}.$$

Thus, $(A, B)$ is controllable iff $(VAV^{-1}, VB)$ is controllable.

A transformation of the form $A \longmapsto VAV^{-1}$ is called a **similarity transformation**; we say $A$ and $VAV^{-1}$ are **similar**.

## Invariance under state feedback

Consider applying the control law $u = Fx + v$ to the system $\dot{x} = Ax + Bu$. Here $F \in \mathbb{R}^{m\times n}$ and $v$ is a new independent input. The new state model is

$$\dot{x} = (A + BF)x + Bv.$$

That is, if $u = Fx + v$ (or $u \longmapsto Fx + u$), then

$$(A, B) \longmapsto (A + BF, B).$$

Notice that the implementation of such a control law requires that there be a sensor for every state variable. To emphasize this, consider the maglev example. Think of the sensors required to implement $u = Fx + v$.

You can prove that under state feedback, the set of reachable states remains unchanged; controllability can be neither created nor destroyed by state feedback.

## Decomposition

If we have a model $\dot{x} = Ax + Bu$ where $(A, B)$ is **not** controllable, it is natural to try to decompose the system into a controllable part and an uncontrollable part. Let's do an example.

**Example** 2 carts, 1 force

We had these matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

And the controllability matrix is

$$W_c = \begin{bmatrix} 0 & -1 & 0 & 3 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 0 & -6 \\ 2 & 0 & -6 & 0 \end{bmatrix}.$$

The rank equals 2. The set of reachable states is the 2-dimensional subspace spanned by the first two columns. Let $\{e_1, e_2\}$ denote these two columns; thus $\{e_1, e_2\}$ is a basis for Im $W_c$. Add two more vectors to get a basis for $\mathbb{R}^4$, say

$$e_3 = (0, 0, 1, 0), \quad e_4 = (0, 0, 0, 1).$$

Now, the matrix $A$ is the matrix representation of an LT in the standard basis. Write the matrix of the same LT but in the basis $\{e_1, \ldots, e_4\}$. As you recall, you proceed as follows: Write $Ae_1$ in the new basis and stack up the coefficients as the first column:

$$Ae_1 = e_2.$$

So the first column of the new matrix is $(0, 1, 0, 0)$. Repeat for the other basis vectors. The result is the matrix

$$\begin{bmatrix} 0 & -3 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There's a more streamlined way to describe this transformation. Form the matrix $V$ by putting $\{e_1, \ldots, e_4\}$ as its columns:

$$V = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Now transform the state via $x = V\tilde{x}$. Then $A, B$ transform to $V^{-1}AV, V^{-1}B$:

$$V^{-1}AV = \begin{bmatrix} 0 & -3 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad V^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

These matrices have a very nice structure, indicated by the partition lines:

$$V^{-1}AV = \left[\begin{array}{cc|cc} 0 & -3 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad V^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right].$$

Let us write the blocks like this:

$$V^{-1}AV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad V^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Thus, the state $x$ has been transformed to $\tilde{x} = V^{-1}x$, and the state equation $\dot{x} = Ax + Bu$ to

$$\begin{aligned} \dot{\tilde{x}}_1 &= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 + B_1 u \\ \dot{\tilde{x}}_2 &= A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + B_2 u. \end{aligned}$$

Note these key features: $B_2 = 0$, $A_{21} = 0$, and $(A_{11}, B_1)$ is controllable. With these, the model is actually

$$\begin{aligned}\dot{\tilde{x}}_1 &= A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 + B_1 u \\ \dot{\tilde{x}}_2 &= A_{22}\tilde{x}_2.\end{aligned}$$

In this form, $\tilde{x}_2$ represents the uncontrollable part of the system—the second equation has no input. And $\tilde{x}_1$ represents the controllable part. There is coupling only from $\tilde{x}_2$ to $\tilde{x}_1$. □

Now we describe the general theory. The matrix $W_c$ is the controllability matrix and the subspace Im $W_c$ is the set of reachable states. It's convenient to rename this subspace as the **controllable subspace**. Now, if we have a model $\dot{x} = Ax + Bu$ where $(A, B)$ is *not* controllable, we can decompose the system into a controllable part and an uncontrollable part.

The decomposition construction follows the example. Let $\{e_1, \dots, e_k\}$ be a basis for Im $W_c$. Complement it to get a full basis for state space:

$$\{e_1, \dots, e_k, \dots, e_n\}.$$

Let $V$ denote the square matrix with columns $e_1, \dots, e_n$. Then

$$V^{-1}AV = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad V^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

Furthermore, $(A_{11}, B_1)$ is controllable. The lower-left block of the new $A$ equals zero because Im $W_c$ is $A$-invariant; the lower block of the new $B$ equals zero because Im $B \subset$ Im $W_c$.

## 4.3 The PBH (Popov-Belevitch-Hautus) Test

If we have the model $\dot{x} = Ax + Bu$ and we want to check if $(A, B)$ is controllable, that is, if there is enough control authority, we can check the rank of $W_c$. This section develops an alternative test that is frequently more insightful.

Observe that the $n \times n$ matrix $A - \lambda I$ is invertible iff $\lambda$ is not an eigenvalue. In other words

$$\text{rank}(A - \lambda I) = n \iff \lambda \text{ is not an eigenvalue.}$$

The PBH test concerns the $n \times (n + m)$ matrix

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix}.$$

**Theorem 4.3.1** *$(A, B)$ is controllable iff*

$$\textit{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \quad \forall \textit{ eigenvalues } \lambda \textit{ of } A.$$

**Proof**

($\Longrightarrow$) Assume rank $\begin{bmatrix} A - \lambda I & B \end{bmatrix} < n$ for some eigenvalue. Then there exists $x \neq 0$ ($x$ will be complex if $\lambda$ is) such that

$$x^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0$$

(i.e., $x \perp$ every column of $\begin{bmatrix} A-\lambda I & B \end{bmatrix}$), where * denotes complex-conjugate transpose. Thus

$$x^* A = \lambda x^*, \ \ x^* B = 0.$$

So

$$x^* A^2 = \lambda x^* A = \lambda^2 x^*$$

etc.

$$\Rightarrow x^* A^k = \lambda^k x^*.$$

Thus

$$x^* \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} x^*B & \lambda x^* B & \cdots & \lambda^{n-1} x^* B \end{bmatrix} = 0.$$

So $(A, B)$ is not controllable.

($\Longleftarrow$) Assume $(A, B)$ is not controllable. As in the preceding section, there exists $V$ such that

$$(\tilde{A}, \tilde{B}) = (V^{-1}AV, V^{-1}B)$$

have the form

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

Thus rank $[\tilde{A} - \lambda I \ \tilde{B}] < n$ for $\lambda$ an eigenvalue of $A_{22}$. So rank $[A - \lambda I \ B] < n$ since

$$\begin{bmatrix} \tilde{A} - \lambda I & \tilde{B} \end{bmatrix} = V^{-1} \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}.$$

$\square$

In view of this theorem, it makes sense to define an **eigenvalue** $\lambda$ of $A$ to be **controllable** if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n.$$

Then $(A, B)$ is controllable iff every eigenvalue of $A$ is controllable.

**Example** 2 carts, 1 force

We had these matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

The eigenvalues of $A$ are $\{0, 0, \pm\sqrt{3}j\}$. The controllable ones are $\{\pm\sqrt{3}j\}$, which are the eigenvalues of $A_{11}$, the controllable part of $A$. $\square$

## 4.4 Controllability from a Single Input

In this section we ask, when is a system controllable from a single input? That is, we consider $(A, B)$ pairs where $A$ is $n \times n$ and $B$ is $n \times 1$.

A matrix of the form

$$\begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{bmatrix}$$

is called a **companion matrix**. Its characteristic polynomial is

$$s^n + a_n s^{n-1} + \cdots + a_2 s + a_1.$$

Companion matrices arise naturally in going from a differential equation model to a state model.

**Example** Consider the system modeled by

$$\dddot{y} + a_3\ddot{y} + a_2\dot{y} + a_1 y = b_3\ddot{u} + b_2\dot{u} + b_1 u.$$

The transfer function is

$$\frac{b_3 s^2 + b_2 s + b_1}{s^3 + a_3 s^2 + a_2 s + a_1}$$

and then the controllable realization is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}, \quad D = 0$$

□

Notice that if $A$ is a companion matrix, then there exists a vector $B$ such that $(A, B)$ is controllable, namely,

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

**Example** The controllability matrix of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is

$$W_c = \left[\begin{array}{ccc} B & AB & A^2B \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & -a_3 \\ 1 & -a_3 & -a_2 + a_3^2 \end{array}\right].$$

The rank of $W_c$ equals 3, so $(A, B)$ is controllable. □

Now we'll see that if $(A, B)$ is controllable and $B$ is $n \times 1$, then $A$ is similar to a companion matrix.

**Theorem 4.4.1** *Suppose $(A, B)$ is controllable and $B$ is $n \times 1$. Let the characteristic polynomial of $A$ be*

$$s^n + a_n s^{n-1} + \cdots + a_1.$$

*Define*

$$\tilde{A} = \left[\begin{array}{ccccc} 0 & 1 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{array}\right], \quad \tilde{B} = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array}\right].$$

*Then there exists a $W$ such that*

$$W^{-1}AW = \tilde{A}, \ W^{-1}B = \tilde{B}.$$

**Proof** Assume $n = 3$ to simplify the notation. The characteristic poly of $A$ is

$$s^3 + a_3 s^2 + a_2 s + a_1.$$

By Cayley-Hamilton,

$$A^3 + a_3 A^2 + a_2 A + a_1 I = 0.$$

Multiply by $B$:

$$A^3 B + a_3 A^2 B + a_2 AB + a_1 B = 0.$$

Hence

$$A^3 B = -a_1 B - a_2 AB - a_3 A^2 B.$$

Using this equation, you can verify that

$$A \left[\begin{array}{ccc} B & AB & A^2B \end{array}\right] = \left[\begin{array}{ccc} B & AB & A^2B \end{array}\right] \left[\begin{array}{ccc} 0 & 0 & -a_1 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_3 \end{array}\right]. \tag{4.1}$$

Define the controllability matrix $W_c := \begin{bmatrix} B & AB & A^2B \end{bmatrix}$ and the new matrix

$$M = \begin{bmatrix} 0 & 0 & -a_1 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_3 \end{bmatrix}.$$

Note that $M^T$ is the companion matrix corresponding to the characteristic polynomial of $A$. From (4.1) we have

$$W_c^{-1} A W_c = M. \tag{4.2}$$

Regarding $B$, we have

$$B = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

so

$$W_c^{-1} B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{4.3}$$

Now define

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

These are the matrices we want to transform $A, B$ to. Define their controllability matrix

$$\tilde{W}_c := \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} \end{bmatrix}.$$

Then, as in (4.2) and (4.3),

$$\tilde{W}_c^{-1} \tilde{A} \tilde{W}_c = M, \ \text{(same } M\text{!)} \tag{4.4}$$

$$\tilde{W}_c^{-1} \tilde{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{4.5}$$

From (4.2), (4.4) and (4.3), (4.5),

$$W_c^{-1} A W_c = \tilde{W}_c^{-1} \tilde{A} \tilde{W}_c, \ \tilde{W}_c^{-1} B = \tilde{W}_c^{-1} \tilde{B}.$$

Define $W = W_c \tilde{W}_c^{-1}$. Then

$$W^{-1} A W = \tilde{A}, \quad W^{-1} B = \tilde{B}.$$

□

Let's summarize the steps to transform $(A, B)$ to $(\tilde{A}, \tilde{B})$ :

**Procedure**

**Step 1** Find the characteristic poly of $A$:

$$s^n + a_n s^{n-1} + \cdots + a_1.$$

**Step 2** Define

$$W_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix},\ \tilde{W}_c = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix},\ W = W_c \tilde{W}_c^{-1}.$$

Then $W^{-1}AW = \tilde{A},\ W^{-1}B = \tilde{B}$.

**Example**

$$A = \begin{bmatrix} 3 & -2 & 9 \\ -2 & 2 & -7 \\ -1 & 1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$$

char poly $A = s^3 - s^2 - 2s + 1$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$W_c = \begin{bmatrix} -3 & -6 & -10 \\ 3 & 5 & 8 \\ 1 & 2 & 3 \end{bmatrix}, \quad \tilde{W}_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$W = \begin{bmatrix} 2 & -3 & -3 \\ -3 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}$$

□

## 4.5 Pole Assignment

This section presents the most important result about controllability: That eigenvalues[1] can be arbitrarily assigned by state feedback. This result is very useful and is the key to state-space control design methods. The result was first proved by W.M. Wonham, a professor in our own ECE department.

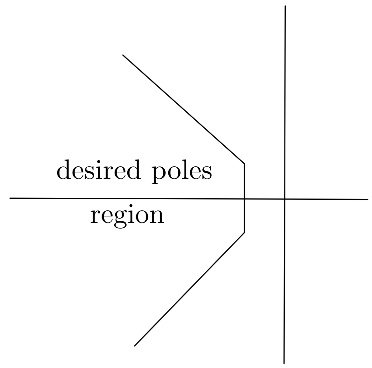
As we saw, the control law $u = Fx + v$ transforms $(A, B)$ to $(A + BF, B)$. This leads us to pose the **pole assignment problem**:

Given $A, B$.

Design $F$ so that the eigs of $A + BF$ are in a desired location.

[1] It is customary to call the eigenvalues of $A$ its "poles." Of course this more properly refers to the poles of the relevant transfer function.

We might like to specify the eigenvalues of $A+BF$ exactly, or we might be satisfied to have them in a specific region in the complex plane, such as a truncated cone for fast transient response and good damping:




The fact is that you can arbitrarily assign the eigs of $A+BF$ iff $(A,B)$ is controllable. We'll prove this, and also get a procedure to design $F$.

### Single-Input Case

**Example**

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This $A$ is in companion form, and also $B$ is as in Theorem 4.4.1. Let us design $F = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}$ to place the eigs of $A+BF$ at $\begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$. We have $A+BF$ is a companion matrix with final row

$$\begin{bmatrix} 1+F_1 & -1+F_2 & -1+F_3 \end{bmatrix}.$$

Thus its characteristic poly is

$$s^3 + (1-F_3)s^2 + (1-F_2)s + (-1-F_1).$$

But the desired char poly is

$$(s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6.$$

Equating coefficients in these two equations, we get the unique $F$:

$$F = -\begin{bmatrix} 7 & 10 & 5 \end{bmatrix}.$$

□

The example extends to general case of $A$ a companion matrix with final row

$$\begin{bmatrix} -a_1 & \cdots & -a_n \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

as follows: Let $\{\lambda_1, \ldots, \lambda_n\}$ be the desired set of eigenvalues, occurring in complex-conjugate pairs. The matrix $F$ has the form $F = \begin{bmatrix} F_1 & \cdots & F_n \end{bmatrix}$, so $A + BF$ is a companion matrix with final row

$$\begin{bmatrix} -a_1 + F_1 & \cdots & -a_n + F_n \end{bmatrix}.$$

Equate the coefficients of the two polynomials

$$s^n + (a_n - F_n)s^{n-1} + \cdots + (a_1 - F_1), \quad (s - \lambda_1) \cdots (s - \lambda_n),$$

and solve for $F_1, \ldots, F_n$.

Let us turn to the general case of $(A, B)$ controllable, $B$ is $n \times 1$. The procedure to compute $F$ to assign the eigenvalues of $A + BF$ is as follows:

**Step 1** Using Theorem 4.4.1, compute $W$ so that

$$\tilde{A} := W^{-1}AW, \; \tilde{B} := W^{-1}B$$

have the form that $\tilde{A}$ is a companion matrix and

$$\tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

**Step 2** Compute $\tilde{F}$ to assign the eigs of $\tilde{A} + \tilde{B}\tilde{F}$ to the desired locations.

**Step 3** Set $F = \tilde{F}W^{-1}$.

To see that $A + BF$ and $\tilde{A} + \tilde{B}\tilde{F}$ have the same eigs, simply note that

$$W^{-1}(A + BF)W = \tilde{A} + \tilde{B}\tilde{F}.$$

**Example** 2 pendula

We had

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}, \quad u = \ddot{d}.$$

Taking $L_1 = 1$, $L_2 = 1/2$, $g = 10$, we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 20 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}.$$

Suppose the desired eigs are $\{-1, -1, -2 \pm j\}$

**Step 1**

$$\text{eigs } A = \left\{\pm\sqrt{10}, \ \pm\sqrt{20}\right\}$$

$$\text{char poly } A = s^4 - 30s^2 + 200$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -200 & 0 & 30 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$W_c = \begin{bmatrix} 0 & -1 & 0 & -10 \\ -1 & 0 & -10 & 0 \\ 0 & -2 & 0 & -40 \\ -2 & 0 & -40 & 0 \end{bmatrix}$$

$$\tilde{W}_c = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 30 \\ 1 & 0 & 30 & 0 \end{bmatrix}$$

$$W = W\tilde{W}_c^{-1} = \begin{bmatrix} 20 & 0 & -1 & 0 \\ 0 & 20 & 0 & -1 \\ 20 & 0 & -2 & 0 \\ 0 & 20 & 0 & -2 \end{bmatrix}$$

**Step 2** Desired char poly is

$$(s+1)^2(s+2-j)(s+2+j) = s^4 + 6s^3 + 14s^2 + 14s + 5.$$

Char poly of $\tilde{A} + \tilde{B}\tilde{F}$ is

$$s^4 - \tilde{F}_4 s^3 + (-30 - \tilde{F}_3)s^2 - \tilde{F}_2 s + (200 - \tilde{F}_1).$$

Equate coeffs and solve:

$$\tilde{F} = \begin{bmatrix} 195 & -14 & -44 & -6 \end{bmatrix}.$$

**Step 3**

$$F = \tilde{F}W^{-1} = \left[\begin{array}{cccc} -24.5 & -7.4 & 34.25 & 6.7 \end{array}\right].$$

This result is the same as via the MATLAB command *acker*. □

We have now proved sufficiency of the **single-input pole assignment theorem**:

**Theorem 4.5.1** *The eigs of $A + BF$ can be arbitrarily assigned iff $(A, B)$ is controllable.*

The direction ($\Longrightarrow$) is left as an exercise. It says, if $(A, B)$ is not controllable, then some eigenvalues of $A + BF$ are fixed, that is, they do not move as $F$ varies.

Note that in the single-input case, $F$ is uniquely determined by the set of desired eigenvalues.

### Multi-Input Case

Now we turn to the general case of $(A, B)$ where $B$ is $n \times m$, $m > 1$. As before, if $(A, B)$ is not controllable, then some of the eigs of $A + BF$ are fixed; if $(A, B)$ is controllable, the eigs can be freely assigned.

To prove this, and construct $F$, we first see that the system can be made controllable from a single input by means of a preliminary feedback.

**Lemma 4.5.1** *(Heymann 1968) Assume $(A, B)$ is controllable. Let $B_1$ be any nonzero column of $B$ (could be the first one). Then $\exists F$ such that $(A + BF, B_1)$ is controllable.*

Before proving the lemma, let's see how it's used.

**Theorem 4.5.2** *(Wonham 1967) The eigs of $A+BF$ are freely assignable iff $(A, B)$ is controllable.*

**Proof** ($\Longrightarrow$) For you to do.

($\Longleftarrow$)

**Step 1** Select, say, the first column $B_1$ of $B$. If $(A, B_1)$ is controllable, set $\tilde{F} = 0$ and go to Step 3.

**Step 2** Choose $\tilde{F}$ so that $(A + B\tilde{F}, B_1)$ is controllable.

**Step 3** Compute $\tilde{\tilde{F}}$ so that $A + B\tilde{F} + B_1\tilde{\tilde{F}}$ has the desired eigs.

**Step 4** Set $F = \tilde{F} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tilde{\tilde{F}}$. Then

$$A + BF = A + B\tilde{F} + B_1\tilde{\tilde{F}}.$$

□

Two comments: A random $\tilde{F}$ will work in Step 2; In general $F$ is not uniquely determined by the desired eigenvalues.

Now we turn to the proof of the lemma.

**Proof** (Hautus '77)

The proof concerns an artificial discrete-time state model, namely,

$$x(k+1) = Ax(k) + Bu(k),\ x(1) = B_1. \tag{4.6}$$

Suppose we can prove this: There exists an input sequence $\{u(1), \ldots, u(n-1)\}$ such that the resulting states $\{x(1), \ldots, x(n)\}$ span $\mathbb{R}^n$, that is, they are linearly independent. Then define $F$ as follows:

$$u(n) = \text{ anything}$$

$$F \begin{bmatrix} x(1) & \cdots & x(n) \end{bmatrix} = \begin{bmatrix} u(1) & \cdots & u(n) \end{bmatrix}. \tag{4.7}$$

To see that $(A + BF, B_1)$ is controllable, note from (4.6) and (4.7) that

$$x(k+1) = (A + BF)x(k),\ x(1) = B_1,$$

so

$$\begin{bmatrix} x(1) & \cdots & x(n) \end{bmatrix} = \begin{bmatrix} B_1 & (A+BF)B_1 & \cdots & (A+BF)^{n-1}B_1 \end{bmatrix}.$$

Thus the controllability matrix of $(A + BF, B_1)$ has rank $n$.

So it suffices to show, with respect to (4.6), that there exist $\{u(1), \ldots, u(n-1)\}$ such that the set $\{x(1), \ldots, x(n)\}$ is lin. indep. This is proved by induction.

First, $\{x(1)\} = \{B_1\}$ is lin. indep. since $B_1 \neq 0$.

For the induction hypothesis, suppose we've chosen $\{u(1), \ldots, u(k-1)\}$ such that $\{x(1), \ldots, x(k)\}$ is lin. indep. Define

$$\mathcal{V} = \operatorname{Im} \begin{bmatrix} x(1) & \cdots & x(k) \end{bmatrix}.$$

If $\mathcal{V} = \mathbb{R}^n$ we're done. So assume

$$\mathcal{V} \neq \mathbb{R}^n. \tag{4.8}$$

We must now show there exists $u(k)$ such that $x(k+1) \notin \mathcal{V}$. (Note that $x(k+1) \notin \mathcal{V}$ is equivalent to $\{x(1), \ldots, x(k+1)\}$ is lin. indep.) That is, we must show (from (4.6))

$$(\exists u(k))\ Ax(k) + Bu(k) \notin \mathcal{V}. \tag{4.9}$$

Suppose, to the contrary, that

$$(\forall u \in \mathbb{R}^m)\ Ax(k) + Bu \in \mathcal{V}.$$

Setting $u = 0$ we get

$$Ax(k) \in \mathcal{V}. \tag{4.10}$$

Then we also get

$$(\forall u \in \mathbb{R}^m)\ Bu \in \mathcal{V}. \tag{4.11}$$

Now we will show that

$$A\mathcal{V} \subset \mathcal{V}, \tag{4.12}$$

i.e.,

$$Ax(i) \in \mathcal{V}, \quad i = 1, \ldots, k.$$

This is true for $i = k$ by (4.10); for $i < k$ we have from (4.6) that

$$Ax(i) = x(i+1) - Bu(i)$$

where $x(i+1) \in \mathcal{V}$ by definition $\mathcal{V}$, and $Bu(i) \in \mathcal{V}$ by (4.11). This proves (4.12).

Using (4.11) and (4.12) we get that $\forall u \in \mathbb{R}^m$

$$\begin{aligned} ABu &\in A\mathcal{V} \subset \mathcal{V} \\ A^2Bu &\in A^2\mathcal{V} \subset A\mathcal{V} \subset \mathcal{V} \\ &\text{etc.} \end{aligned}$$

Thus every column of the controllability matrix of $(A, B)$ belongs to $\mathcal{V}$. Hence Im $W_c \subset \mathcal{V}$. But Im $W_c = \mathbb{R}^n$, by controllability, so $\mathcal{V} = \mathbb{R}^n$. This contradicts (4.8), so (4.9) is true after all.

□

Finally, the following test for controllability is fairly good numerically:

1. compute the eigs of $A$
2. choose $F$ at random
3. compute the eigs of $A + BF$

Then $(A, B)$ is controllable $\iff$ {eigs $A$} and {eigs $A + BF$} are disjoint.

## 4.6 Stabilizability

Recall that $A$ is **stable** if all its eigenvalues have negative real parts. This is the same as saying that for $\dot{x} = Ax$, $x(t) \to 0$ as $t \to \infty$ for every $x(0)$. Under state feedback, $u = Fx + v$, the system $\dot{x} = Ax + Bu$ is transformed to

$$\dot{x} = (A + BF)x + Bv.$$

We say $(A, B)$ is **stabilizable** if $\exists F$ such that $A + BF$ is stable. By the pole assignment theorem $(A, B)$ controllable $\implies$ $(A, B)$ stabilizable, i.e., controllability is sufficient (a stronger property). What exactly is a test for stabilizability? The following theorem answers this.

**Theorem 4.6.1** *The following three conditions are equivalent:*

1. *$(A, B)$ is stabilizable*
2. *The uncontrollable part of $A$, $A_{22}$, is stable.*
3. *rank* $\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$ *for every eigenvalue* $\lambda$ *of* $A$ *with Re* $\lambda \geq 0$.

The 2 carts, 1 force example is not stabilizable, because the uncontrollable eigenvalues, $0, 0$, aren't stable.

Suppose $(A, B)$ is stabilizable but not controllable, and we want to find an $F$ to stabilize $A+BF$. The way is clear: Transform to see the controllable part and stabilize that. Here are the details: Let $\{e_1, \ldots, e_k\}$ be a basis for Im $W_c$. Complement it to get a full basis for state space:

$$\{e_1, \ldots, e_k, \ldots, e_n\}.$$

Let $V$ denote the square matrix with columns $e_1, \ldots, e_n$. Then

$$V^{-1}AV = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad V^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where $(A_{11}, B_1)$ is controllable. Choose $F_1$ to stabilize $A_{11} + B_1 F_1$. Then

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 F_1 & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

is stable. Transform the feedback matrix back to the original coordinates:

$$F = \begin{bmatrix} F_1 & 0 \end{bmatrix} V^{-1}.$$

## 4.7 Problems

1. Write a Scilab/MATLAB program to verify the Cayley-Hamilton theorem. Run it for

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}.$$

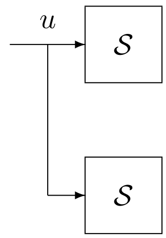
2. Consider the state model $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} -3 & 2 & 2 \\ -1 & 0 & 1 \\ -5 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ -3 & 2 \end{bmatrix}.$$

   (a) Find a basis for the controllable subspace.
   (b) Is the vector $(1, 1, 0)$ reachable from the origin?
   (c) Find a nonzero vector that is orthogonal to every vector that is reachable from the origin.

3. Show that the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has the property that $(A, B)$ is controllable for all nonzero $2 \times 1$ matrices $B$.

4. Consider the setup of two identical systems with a common input:




(An example is two identical pendula balanced on one hand.) Let the upper and lower systems be modeled by, respectively,

$$\dot{x}_1 = Ax_1 + Bu, \quad \dot{x}_2 = Ax_2 + Bu.$$

Assume $(A, B)$ is controllable.

(a) Find a state model for the overall system, taking the state to be $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(b) Find a basis for the controllable subspace of the overall system.

5. Consider the state model $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ -2 & 1 \end{bmatrix}.$$

(a) Transform $x$, $A$, and $B$ so that the controllable part of the system is exhibited explicitly.

(b) What are the controllable eigenvalues?

6. Consider the system with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(a) What are the eigenvalues of $A$?

(b) It is desired to design a state-feedback matrix $F$ to place the eigenvalues of $A + BF$ at $\{-1, -1, -2 \pm j\}$. Is this possible? If so, do it.

(c) It is desired to design a state-feedback matrix $F$ to place the eigenvalues of $A + BF$ at $\{0, 0, -2 \pm j\}$. Is this possible? If so, do it.

7. Consider a pair $(A, B)$ where $A$ is $n \times n$ and $B$ is $n \times 1$ (single input). Show that $(A, B)$ cannot be controllable if $A$ has two linearly independent eigenvectors for the same eigenvalue.

8. Consider the two pendula. Find the controllable subspace when the lengths are equal and when they're not.

9. Continue with the two-pendula problem. Give numerical values to $L_1 \neq L_2$ and stabilize by state feedback.

10. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -1.5 & 1.5 & -1.5 \\ 1 & 2.5 & -0.5 & 2.5 \\ 1 & 1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

   (a) Check that $(A, B)$ is controllable but that $(A, B_i)$ is not controllable for either column $B_i$ of $B$.

   (b) Compute an $F$ to assign the eigenvalues of $A + BF$ to be $-1 \pm j$, $-2 \pm j$.

11. Let $A$ be an $n \times n$ real matrix in companion form. Then $(A, B)$ is controllable for a certain column vector $B$. What does this imply about the Jordan form of $A$?

12. Consider the system model $\dot{x} = Ax + Bu$, $x(0) = 0$ with

$$A = \begin{bmatrix} -3 & -2 & -1 \\ 11 & 6 & 2 \\ -9 & -5 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

   Does there exist an input such that $x(1) = (1, -1, 1)$?

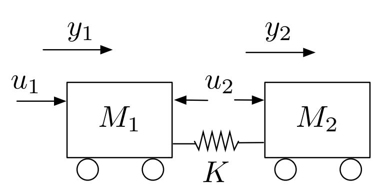
13. Consider the system model $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

   (a) It is desired to design a state feedback matrix $F$ so that the eigenvalues of $A + BF$ all have negative real part. Is this possible?

   (b) Is it possible to design $F$ so that each eigenvalue $\lambda$ of $A + BF$ satisfies Re $\lambda \leq -4$?

14. Consider the following 2-cart system:




   There are two inputs: a force $u_1$ applied to the first cart; a force $u_2$ applied to the two carts in opposite directions as shown. Take the state variables to be

   $$y_1, y_2, \dot{y}_1, \dot{y}_2$$

   *in that order*, and take $M_1 = 2, M_2 = 1, K = 2$.

(a) Find $(A, B)$ in the state model.

(b) According to the Cayley-Hamilton theorem, $A^4$ can be expressed as a linear combination of lower powers of $A$. Derive this expression for the 2-cart system.

15. Many mechanical systems can be modeled by the equation

$$M\ddot{q} + D\dot{q} + Kq = u,$$

where $q$ is a vector of positions (such as joint angles on a robot), $u$ is a vector of inputs, $M$ is a symmetric positive definite matrix, and $D$ and $K$ are two other square matrices. Find a state model by taking $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$. What can you say about controllability of this state model?

16. (a) Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Find $F$ so that the eigenvalues of $A + BF$ are

$$-1 \pm j, -2 \pm j.$$

(b) Take the same $A$ but

$$B = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 6 \end{bmatrix}.$$

Find the controllable part of the system.

17. Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

Is the vector $(4, 1, 4)$ reachable from the origin?

18. Is the following $(A, B)$ pair controllable?

$$A = \begin{bmatrix} 0 & -6 & 0 & 4 \\ 1 & 4 & -1 & 0 \\ 1 & 8 & 0 & 0 \\ 1 & 11 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

19. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(i) Check that $(A, B)$ is controllable.

(ii) Find a feedback law $u = Fx$ such that the closed-loop poles are all at $-1$.

20. Prove that $(A, B)$ is controllable if and only if $(A + BF, B)$ is controllable for some $F$. Prove that $(A, B)$ is controllable if and only if $(A + BF, B)$ is controllable for every $F$.