Final-Value Theorem

Pole locations

Pole locations of $F(s)$ affect qualitative behavior of $f(t)$

When is $f(t)$ bounded, i.e. $|f(t)| < M$ for some M and all $t \geq 0$

When does $f(t)$ have a final value, i.e. $\lim_{t\to\infty} f(t)$ exists

If $\lim_{t\to\infty} f(t)$ exists, what is the value of this limit

Strictly proper, proper, improper

Consider $F(s)$ is rational, i.e. $F(s) = \frac{N(s)}{D(s)}$.

$F(s)$ is *strictly proper* if degree $N(s) <$ degree $D(s)$

$F(s)$ is *proper* if degree $N(s) \leq$ degree $D(s)$

$F(s)$ is *improper* if degree $N(s) >$ degree $D(s)$

We will assume $F(s)$ is strictly proper.

Example

By partial fraction expansion, $F(s)$ can be uniquely written as

$F(s) = G_1(s) + G_2(s) + G_3(s)$

Ex. $G_1(s) = \frac{1}{s+1} + \frac{2}{(s+2)^2+20}$

$G_2(s) = \frac{2}{s} + \frac{4}{s^2}$

$G_3(s) = \frac{1}{s^2+1} + \frac{2}{s-10}$

$f(t) = g_1(t) + g_2(t) + g_3(t)$

$g_1(t) = \mathrm{e}^{-t} + \frac{2}{\sqrt{20}}\mathrm{e}^{-2t}\sin(\sqrt{20}t)$

$g_2(t) = 2 + 4t$

$g_3(t) = \sin t + 2\mathrm{e}^{10t}$

$f(t)$ is unbounded

Example

By partial fraction expansion, $F(s)$ can be uniquely written as

$F(s) = G_1(s) + G_2(s) + G_3(s)$

Ex. $G_1(s) = \frac{1}{s+1} + \frac{2}{(s+2)^2+20}$

$G_2(s) = \frac{2}{s}$

$G_3(s) = 0$

$f(t) = g_1(t) + g_2(t) + g_3(t)$

$g_1(t) = \mathrm{e}^{-t} + \frac{2}{\sqrt{20}}\mathrm{e}^{-2t}\sin(\sqrt{20}t)$

$g_2(t) = 2$

$g_3(t) = 0$

$f(t)$ is bounded
and has a final value 2

Pole locations vs. boundedness

Suppose $F(s)$ is rational and strictly proper.

1. If $F(s)$ has no poles in $\mathrm{Re}(s) \geq 0$, then $f(t)$ is bounded.

2. If $F(s)$ has no poles in $\mathrm{Re}(s) \geq 0$ except a simple pole $s = 0$, and/or some simple complex-conjugate pairs of poles at $\mathrm{Re}(s) = 0$ then $f(t)$ is bounded.

3. In all other cases, $f(t)$ is unbounded.

Pole locations vs. existence of final value

Suppose $F(s)$ is rational and strictly proper.

1. If $F(s)$ has no poles in $\mathrm{Re}(s) \geq 0$, then $f(t)$ has a final value 0.

2. If $F(s)$ has no poles in $\mathrm{Re}(s) \geq 0$ except a simple pole $s = 0$, then $f(t)$ has a final value, $\lim_{s \to 0} sF(s)$.

3. In all other cases, $f(t)$ does not have a final value.

Final value theorem

Suppose $F(s)$ is rational and strictly proper.

Suppose $F(s)$ has no poles in $\mathrm{Re}(s) \geq 0$
except possibly a simple pole $s = 0$.

Then $f(t)$ has a final value, which is

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Example

By partial fraction expansion, $F(s)$ can be uniquely written as
$F(s) = G_1(s) + G_2(s) + G_3(s)$

Ex. $G_1(s) = \frac{1}{s+1} + \frac{2}{(s+2)^2+20}$
$G_2(s) = \frac{2}{s}$
$G_3(s) = 0$

$f(t)$ has a final value:

$$\begin{aligned}\lim_{t\to\infty} f(t) &= \lim_{s\to 0} s(G_1(s) + G_2(s) + G_3(s)) \\ &= \lim_{s\to 0}\left(\frac{s}{s+1} + \frac{2s}{(s+2)^2+20} + 2\right) \\ &= 2\end{aligned}$$