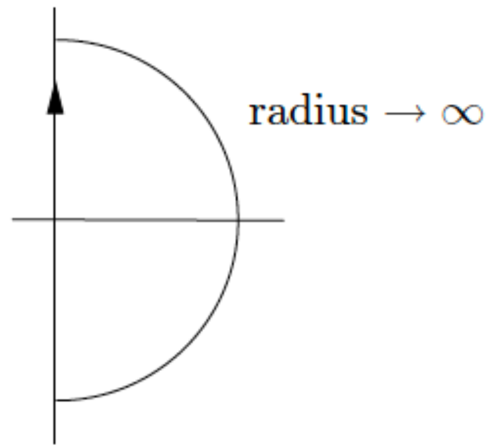
Nyquist Criterion (examples)

Nyquist criterion

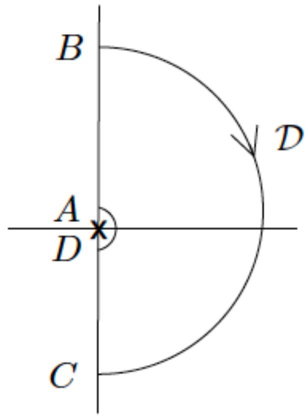


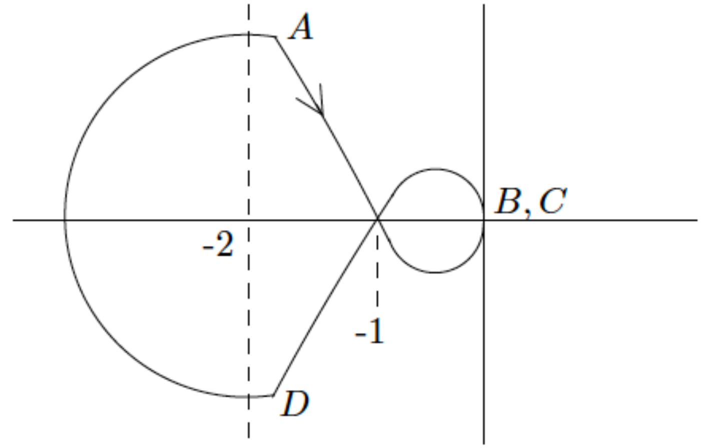
Suppose that $G(s) = P(s)C(s)$ has no poles on the Nyquist contour $\mathcal{D}$ (indenting to the right if necessary), and has n poles in $\mathrm{Re}(s) > 0$

Feedback stability $\Leftrightarrow$
the Nyquist plot $\mathcal{G}$ (i) does not pass through the $-\frac{1}{K}$ and (ii) encircles $-\frac{1}{K}$ exactly n times CCW

(Note: if $n = 0$, then (ii) does not encircle $-\frac{1}{K}$)

Example





$G(s) = P(s)C(s) = \frac{s+1}{s(s-1)}$

Divide Nyquist contour $\mathcal{D}$ into 4 segments:

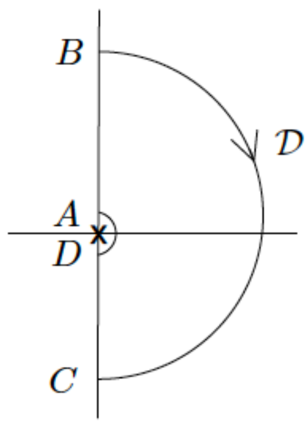
Segment from A to B: $s = j\omega$, ω from ε to ∞

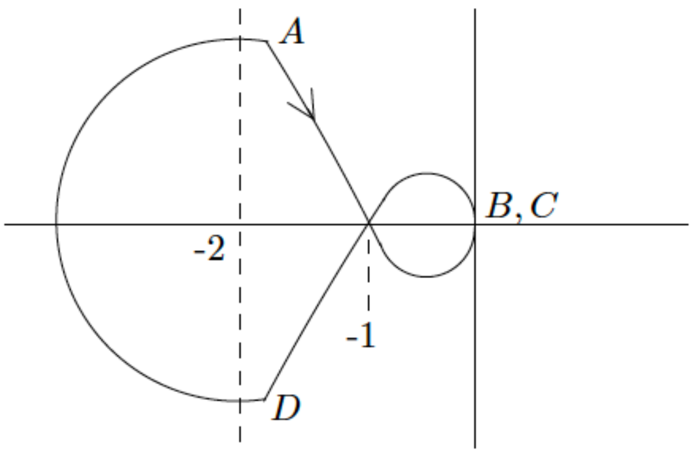
$P(j\omega)C(j\omega) = \frac{j\omega+1}{j\omega(j\omega-1)}$

$\mathrm{Re}(P(j\omega)C(j\omega)) = -\frac{2}{\omega^2+1}$, $\mathrm{Im}(P(j\omega)C(j\omega)) = \frac{1-\omega^2}{\omega(\omega^2+1)}$

$\mathrm{Re}(P(j\varepsilon)C(j\varepsilon)) = -\frac{2}{\varepsilon^2+1} \approx -2$, $\mathrm{Im}(P(j\varepsilon)C(j\varepsilon)) = \frac{1-\varepsilon^2}{\varepsilon(\varepsilon^2+1)} \approx \infty$

Example





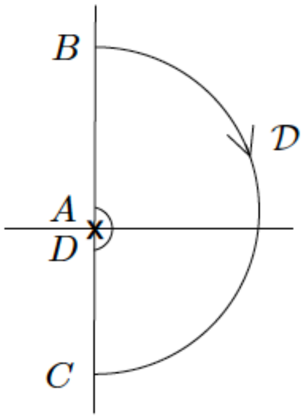
$G(s) = P(s)C(s) = \frac{s+1}{s(s-1)}$

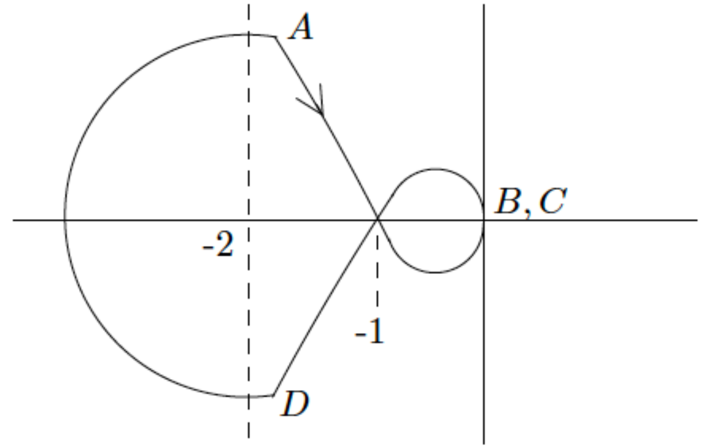
Divide Nyquist contour $\mathcal{D}$ into 4 segments:

Segment from B to C: radius is ∞

$P(j\infty)C(j\infty) = 0$, $P(-j\infty)C(-j\infty) = 0$

Example





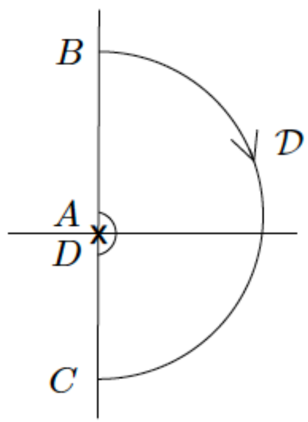
$G(s) = P(s)C(s) = \frac{s+1}{s(s-1)}$

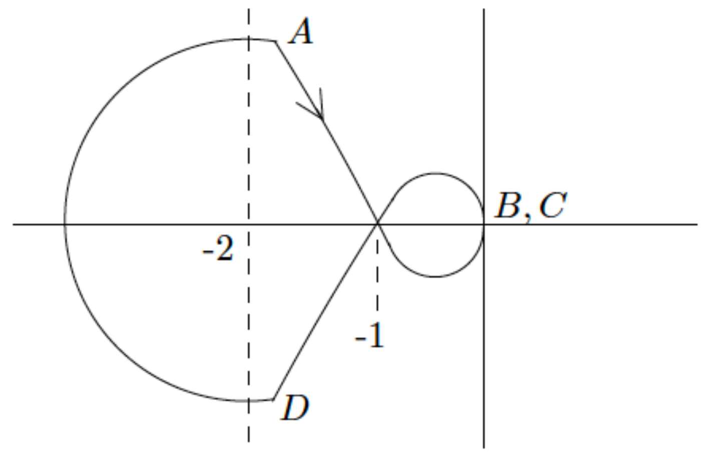
Divide Nyquist contour $\mathcal{D}$ into 4 segments:

Segment from C to D: $s = j\omega$, ω from $-\infty$ to $-\varepsilon$

complex conjugate of the segment from A to B

Example





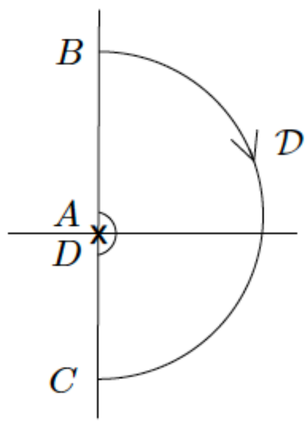
$G(s) = P(s)C(s) = \frac{s+1}{s(s-1)}$

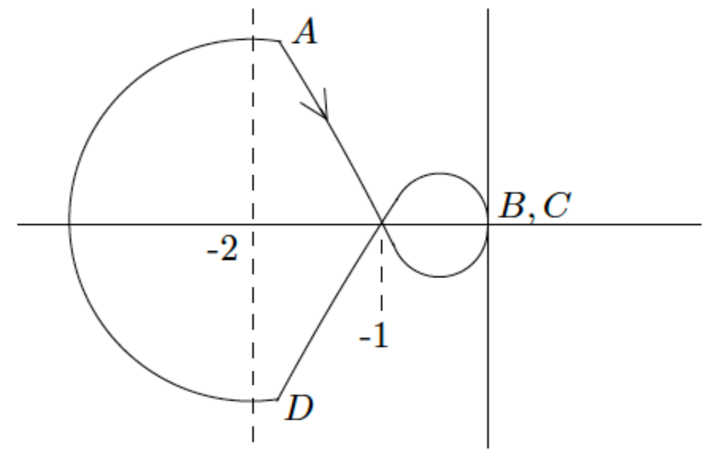
Divide Nyquist contour $\mathcal{D}$ into 4 segments:

Segment from D to A: $s = \varepsilon \mathrm{e}^{j\theta}$, θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$P(s)C(s) = \frac{\varepsilon \mathrm{e}^{j\theta}+1}{\varepsilon \mathrm{e}^{j\theta}(\varepsilon \mathrm{e}^{j\theta}-1)}$$
$$\approx -\frac{1}{\varepsilon \mathrm{e}^{j\theta}}$$

Example





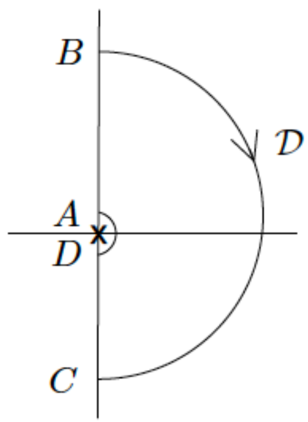
$G(s) = P(s)C(s) = \frac{s+1}{s(s-1)}$

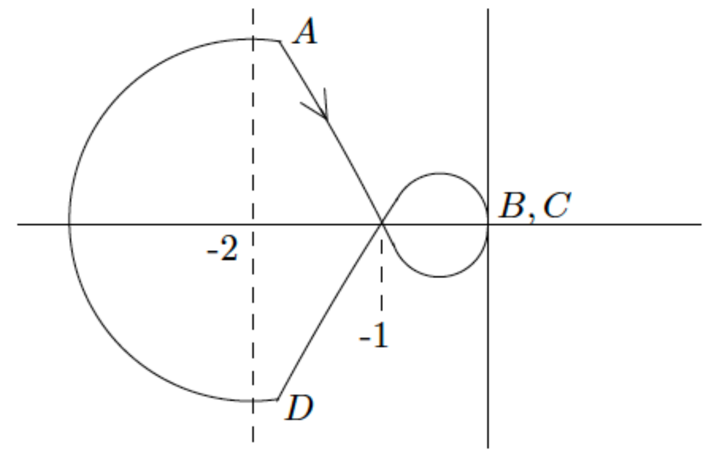
$P(s)C(s)$ has one pole $s = 1$ encircled by $\mathcal{D}$, i.e. $n = 1$

By Nyquist criterion: feedback loop is stable iff the Nyquist plot $\mathcal{G}$ (i) does not pass through $-\frac{1}{K}$ and (ii) encircles $-\frac{1}{K}$ exactly 1 time CCW

So $-1 < -\frac{1}{K} < 0$, i.e. $K > 1$

Example

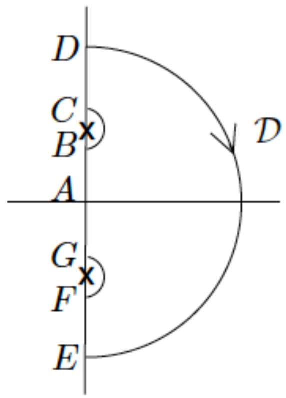


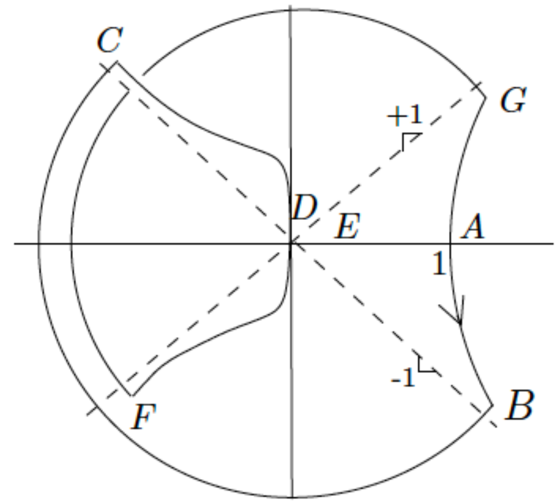


Nyquist plot cannot be drawn by Matlab as a closed contour because Matlab does not indent $\mathcal{D}$ to the right to avoid pole at $s = 0$

So if you use Matlab, you must close the contour to be able to count encirclements

Example





$G(s) = P(s)C(s) = \frac{1}{(s+1)(s^2+1)}$

Divide Nyquist contour $\mathcal{D}$ into 7 segments:

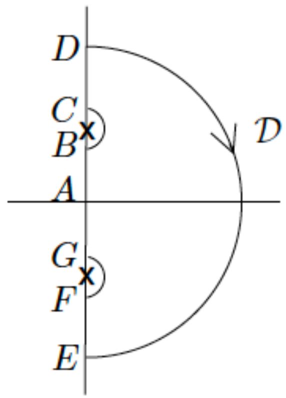
Segment from A to B: $s = j\omega$, ω from 0 to $1 - \varepsilon$

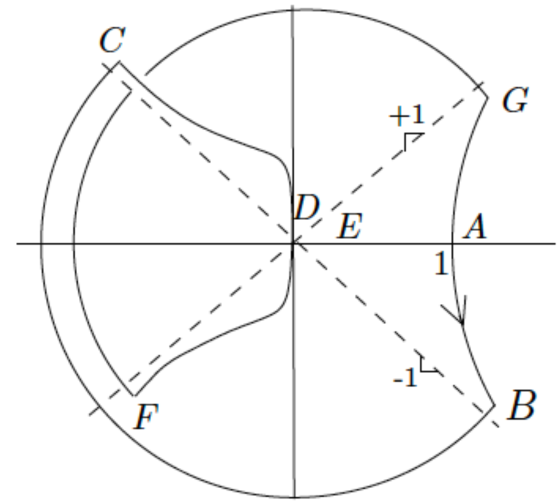
$P(j\omega)C(j\omega) = \frac{1}{(j\omega+1)((j\omega)^2+1)}$

$\mathrm{Re}(P(j\omega)C(j\omega)) = \frac{1}{1-\omega^4}$, $\mathrm{Im}(P(j\omega)C(j\omega)) = \frac{-\omega}{1-\omega^4}$

$\mathrm{Re}(P(j(1-\varepsilon))C(j(1-\varepsilon)) \approx$ $\qquad$ $\mathrm{Im}(P(j(1-\varepsilon))C(j(1-\varepsilon))) \approx$

Example





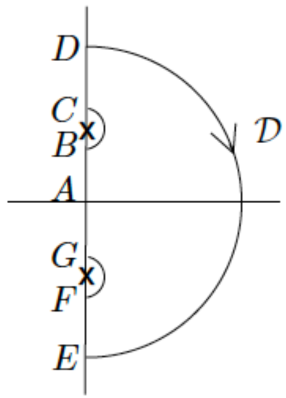
$G(s) = P(s)C(s) = \frac{1}{(s+1)(s^2+1)}$

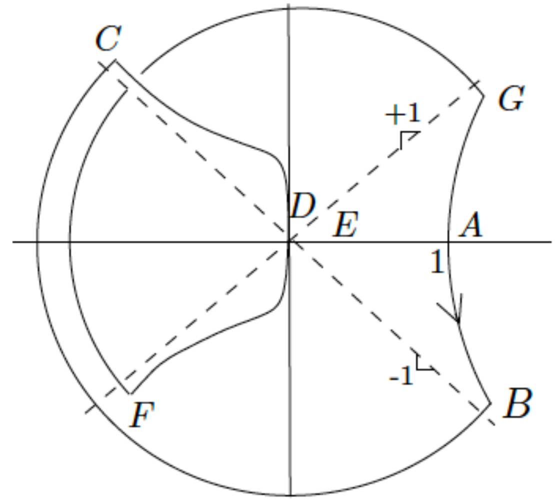
Divide Nyquist contour $\mathcal{D}$ into 7 segments:

Segment from B to C: $s = \varepsilon e^{j\theta} + j$, θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$P(s)C(s) = \frac{1}{(\varepsilon e^{j\theta}+j+1)((\varepsilon e^{j\theta}+j)^2+1)}$

Example





$G(s) = P(s)C(s) = \frac{1}{(s+1)(s^2+1)}$

Divide Nyquist contour $\mathcal{D}$ into 7 segments:

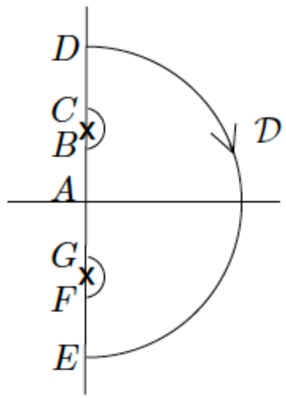
Segment from C to D: $s = j\omega$, ω from $1+\varepsilon$ to ∞

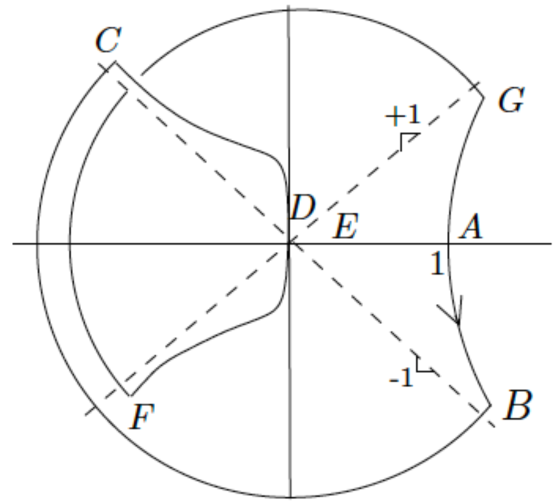
$P(j\omega)C(j\omega) = \frac{1}{(j\omega+1)((j\omega)^2+1)}$

$\text{Re}(P(j\omega)C(j\omega)) = \frac{1}{1-\omega^4}$, $\text{Im}(P(j\omega)C(j\omega)) = \frac{-\omega}{1-\omega^4}$

$\text{Re}(P(j(1+\varepsilon))C(j(1+\varepsilon)) \approx$ $\qquad$ $\text{Im}(P(j(1+\varepsilon))C(j(1+\varepsilon))) \approx$

Example





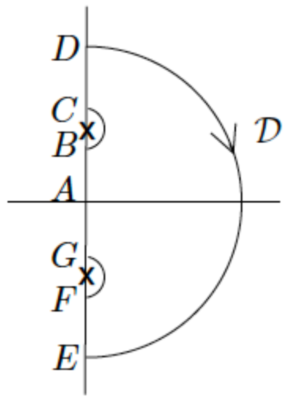
$G(s) = P(s)C(s) = \frac{1}{(s+1)(s^2+1)}$

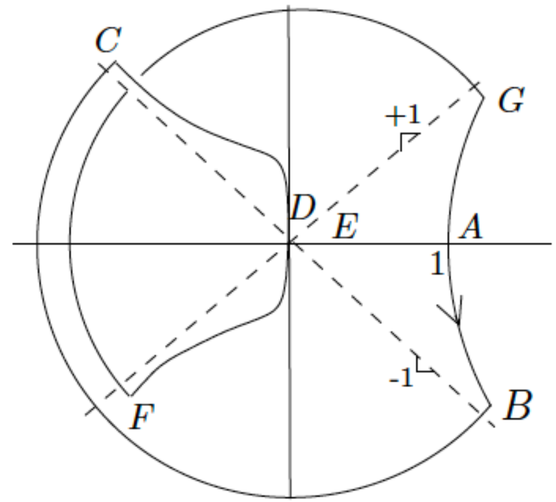
Divide Nyquist contour $\mathcal{D}$ into 7 segments:

Segment from D to E: radius is ∞

$P(j\infty)C(j\infty) = 0$, $P(-j\infty)C(-j\infty) = 0$

Example





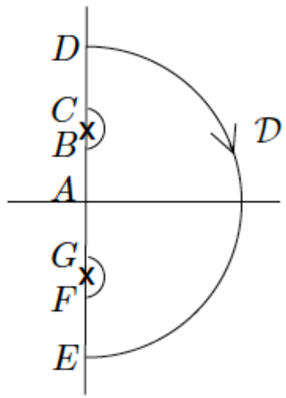
$G(s) = P(s)C(s) = \frac{1}{(s+1)(s^2+1)}$

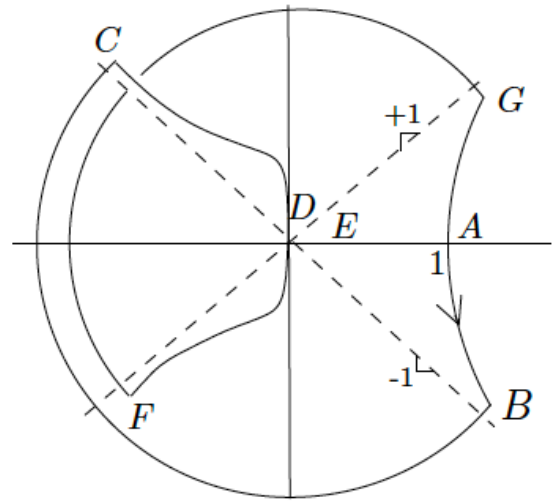
Divide Nyquist contour $\mathcal{D}$ into 7 segments:

Segment from E to F: $s = j\omega$, ω from $-\infty$ to $-1-\varepsilon$

complex conjugate of the segment from C to D

Example





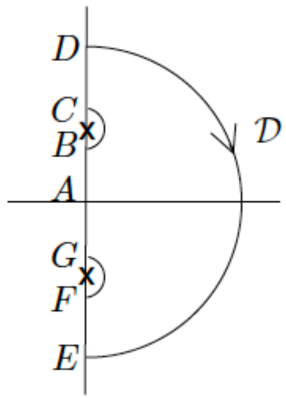
$G(s) = P(s)C(s) = \frac{1}{(s+1)(s^2+1)}$

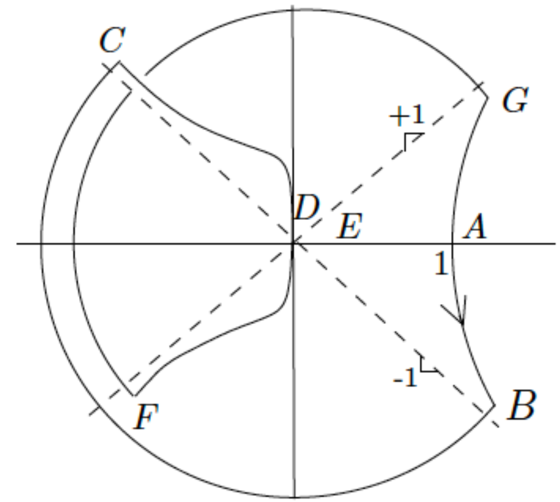
Divide Nyquist contour $\mathcal{D}$ into 7 segments:

Segment from F to G: $s = \varepsilon e^{j\theta} - j$, θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

complex conjugate of the segment from B to C

Example





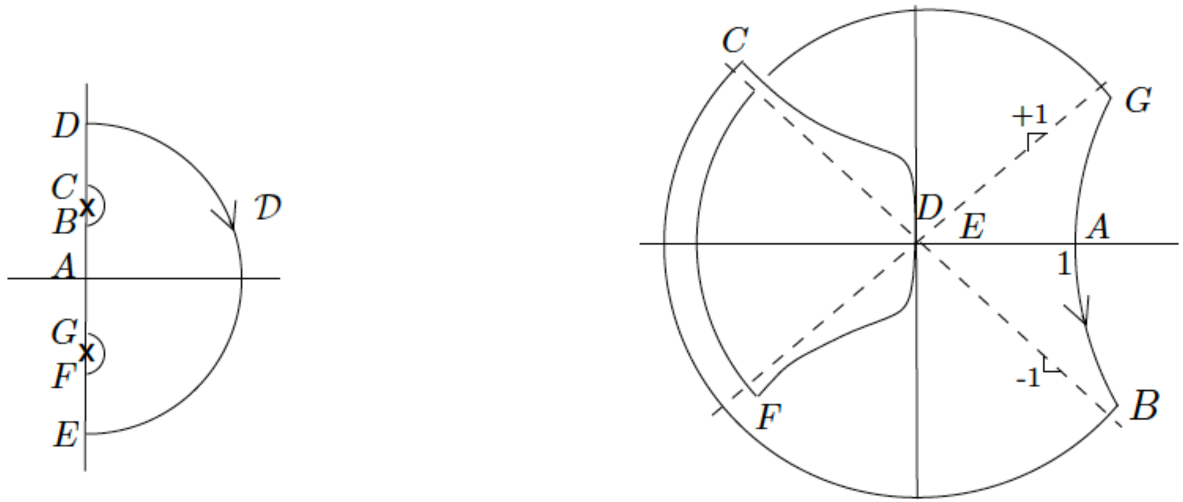
$G(s) = P(s)C(s) = \frac{1}{(s+1)(s^2+1)}$

Divide Nyquist contour $\mathcal{D}$ into 7 segments:

Segment from G to A: $s = j\omega$, ω from $-1+\varepsilon$ to 0

complex conjugate of the segment from A to B

Example



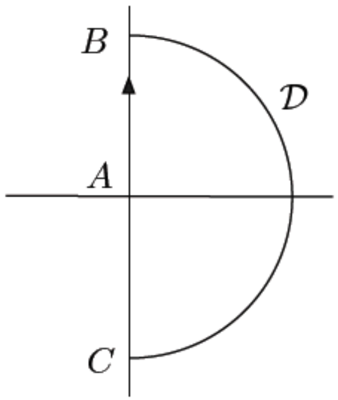
$G(s) = P(s)C(s) = \frac{1}{(s+1)(s^2+1)}$

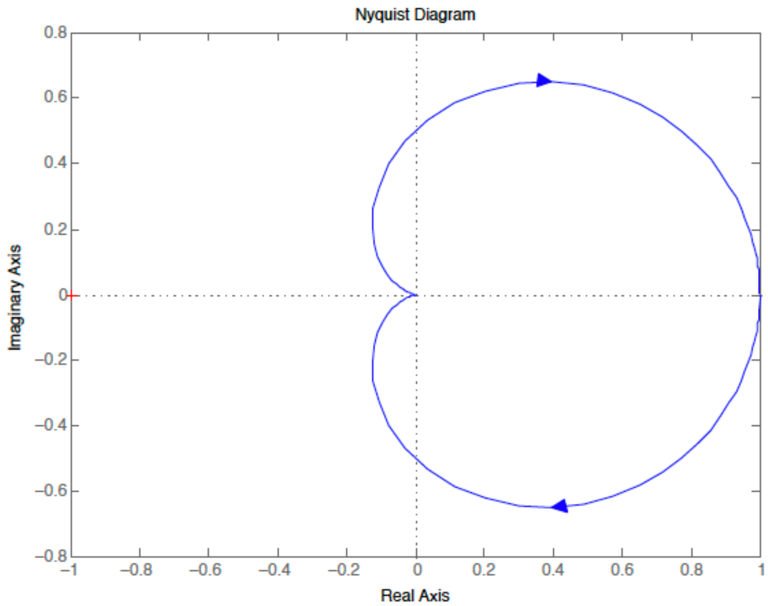
$P(s)C(s)$ has no poles encircled by $\mathcal{D}$, i.e. $n = 0$

By Nyquist criterion: feedback loop is stable iff the Nyquist plot $\mathcal{G}$ (i) does not pass through $-\frac{1}{K}$ and (ii) does not encircle $-\frac{1}{K}$

So $-\frac{1}{K} > 1$, i.e. $-1 < K < 0$

Example





$G(s) = P(s)C(s) = \frac{1}{(s+1)^2}$

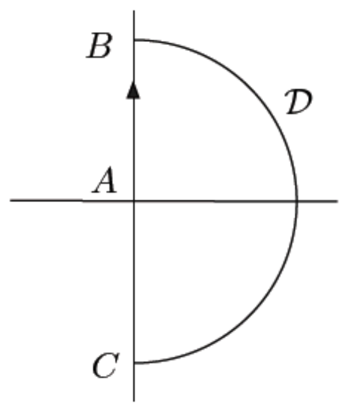
$P(s)C(s)$ has no poles in $\mathrm{Re}(s) > 0$, i.e. $n = 0$

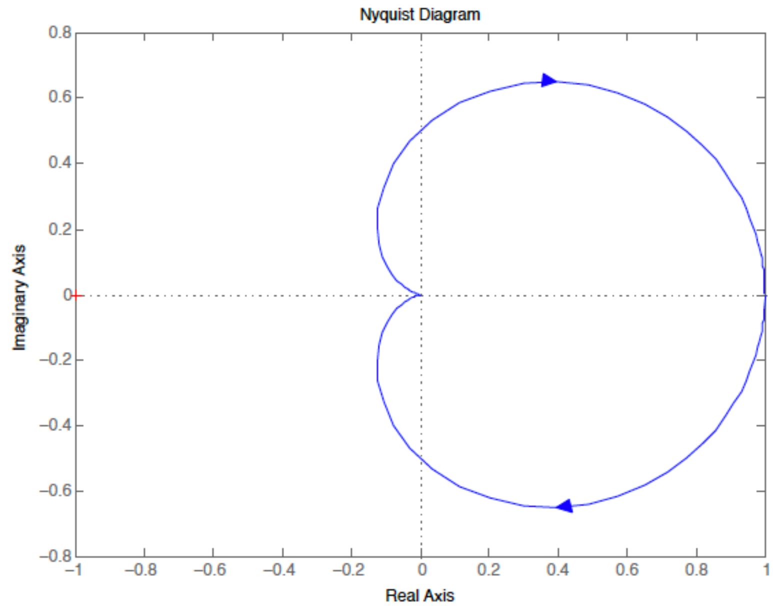
By Nyquist criterion: feedback loop is stable iff the Nyquist plot $\mathcal{G}$ (i) does not pass through $-\frac{1}{K}$ and (ii) does not encircle $-\frac{1}{K}$

So either $-\frac{1}{K} < 0$ or $-\frac{1}{K} > 1$, i.e. either $K > 0$ or $-1 < K < 0$

But $K = 0$ is also fine; so $K > -1$ after all

Minimum phase





$G(s) = P(s)C(s) = \frac{1}{(s+1)^2}$

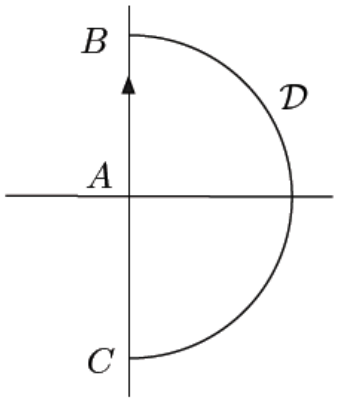
$P(s)C(s)$ has no poles in $\mathrm{Re}(s) > 0$, i.e. $n = 0$

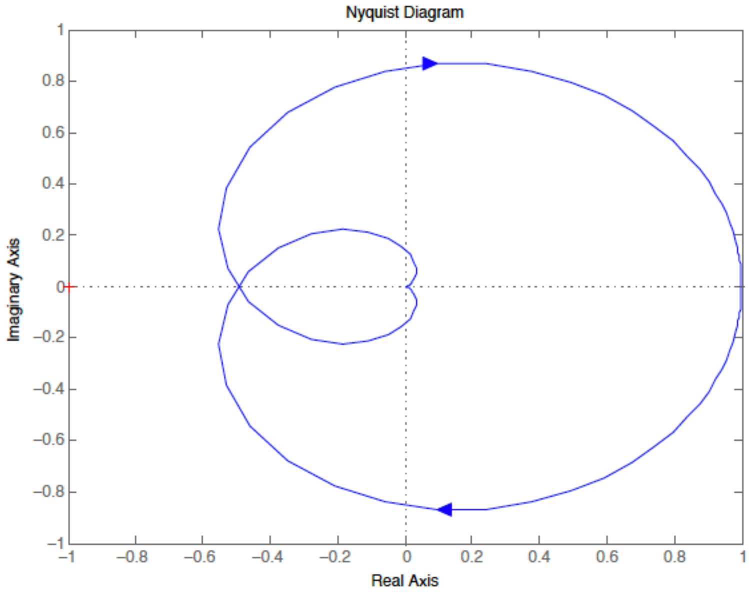
We've seen that closed-loop is stable for all gain $K > -1$

Note: $P(s)C(s)$ has no zeros in the right half-plane

We say $P(s)C(s)$ is *minimum phase* if it has no zeros in the right half-plane

Non minimum phase





$$G(s) = P(s)C(s) = \frac{1}{(s+1)^2}\frac{s-1}{s+1}$$

This $P(s)C(s)$ is non-minimum phase

You can check by Nyquist criterion that closed-loop is stable for all gain $-1 < K < 2$

Non-minimum phase $P(s)C(s)$ is generally harder to be made stable than the analogous minimum phase $P(s)C(s)$