Chapter 8

Affine Formation in Arbitrary Dimensional Space

In this chapter, we study a formation control problem of multi-agent systems in arbitrary dimensional space. In Chapter 6 we introduced a similar formation control problem in 2D, which is applicable to teams of autonomous robots and mobile sensors moving on a plane. However, applications such as formation flying of unmanned aerial vehicles and ocean data retrieval of autonomous underwater vehicles, 3D formation control methods are needed.

This chapter introduces a new formation control problem called *affine formation control*, which includes Chapter 6's 2D similar formation control as a special case. Specifically, in a d (≥ 2) dimensional space, a network of agents is required to form a geometric shape, which can be obtained from a prescribed desired shape via translation, rotation, and *dimension-wise scaling*. The dimension-wise scaling means that scaling factors along each dimension are possibly different. Precisely when all dimensions have identical scaling factors, affine formation control coincides with similar formation control.

The solution for similar formation control in Chapter 6 was based on complex Laplacian, which is however restricted to 2D only. To solve affine formation control in arbitrary dimensions, we introduce the third type of graph Laplacian: *signed Laplacian*. Modeling the interacting agents by digraphs, we show that a necessary graphical condition to achieve affine formation in a d (≥ 2) dimensional space is that the digraph contains a *spanning* $(d+1)$*-tree*, namely there exists (at least) $d+1$ agents that can reach all the other agents through independent paths. These $d+1$ root agents play the role of *leaders*, which determine the translation, rotation, and dimension-wise scaling offsets from the prescribed shape. Under this graphical condition, we present a distributed algorithm for the agents to achieve affine formations in arbitrary dimensions.

8.1 Problem Statement

Consider a network of n (> 1) agents in a d (≥ 2) dimensional space. Each agent i $(\in [1, n])$ has a *state* variable $x_i(t) \in \mathbb{R}^d$, which is a d-dimensional real vector and denotes the position of agent i

in the d-dimensional space at time t. The time $t \geq 0$ is a (nonnegative) real number and denotes the *continuous* time. The motion of each agent is governed by the following ordinary differential equation:

$$\dot{x}_i = u_i, \quad i \in [1, n] \tag{8.1}$$

where $u_i(t) \in \mathbb{R}^d$ is the d-dimensional control input.

Let digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ model the interconnection structure of the n agents. Each *node* in $\mathcal{V} = \{1, ..., n\}$ stands for an agent, and each directed *edge* (j, i) in $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes that agent i can measure the *relative position* of agent j (namely $x_j - x_i$ in agent i's coordinate frame). The *neighbor set* of agent i is $\mathcal{N}_i := \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$.

Moreover, consider that digraph $\mathcal{G}$ is weighted: each edge $(j, i) \in \mathcal{V}$ is associated with a real-valued weight $a_{ij} \in \mathbb{R}$. Hence the adjacency matrix $A = (a_{ij})$, degree matrix $D = \mathrm{diag}(A\mathbf{1}_n)$, and Laplacian matrix $L = D - A$ are all real. Note that the adjacency matrix A is not a nonnegative matrix in general; thus L is a *signed Laplacian matrix*.

Define a *target configuration*

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \mathbb{R}^{nd}, \quad \text{where } \xi_i \in \mathbb{R}^d \text{ and } i \in [1, n]$$

to be the assignment of the n agents to (d-dimensional) points in a global coordinate frame Σ. This configuration ξ specifies the d-dimensional formation *shape* that the agents are required to achieve. To consider not just the 'consensus formation', we henceforth assume that ξ is linearly independent from $\mathbf{1}_{nd}$ (the vector of nd ones).

Given a target configuration $\xi \in \mathbb{R}^{nd}$, we say that another configuration $\xi' \in \mathbb{R}^{nd}$ is *affine* to ξ if there exist a matrix $A \in \mathbb{R}^{d \times d}$ and a vector $a \in \mathbb{R}^d$ such that

$$(\forall i \in [1, n])\xi'_i = A\xi_i + a.$$

Since an arbitrary real matrix A may be factorized by *singular value decomposition* as $A = U\Gamma V$, where U, V are *unitary matrices* (i.e. $UU^\top = U^\top U = I, VV^\top = V^\top V = I$) and Γ is a $d \times d$ diagonal matrix (diagonal entries being singular values), configuration ξ' can be obtained from ξ via a rotation by V, a scaling along every dimension by Γ, another rotation by U, and finally a translation by a. This is an *affine motion* from ξ.

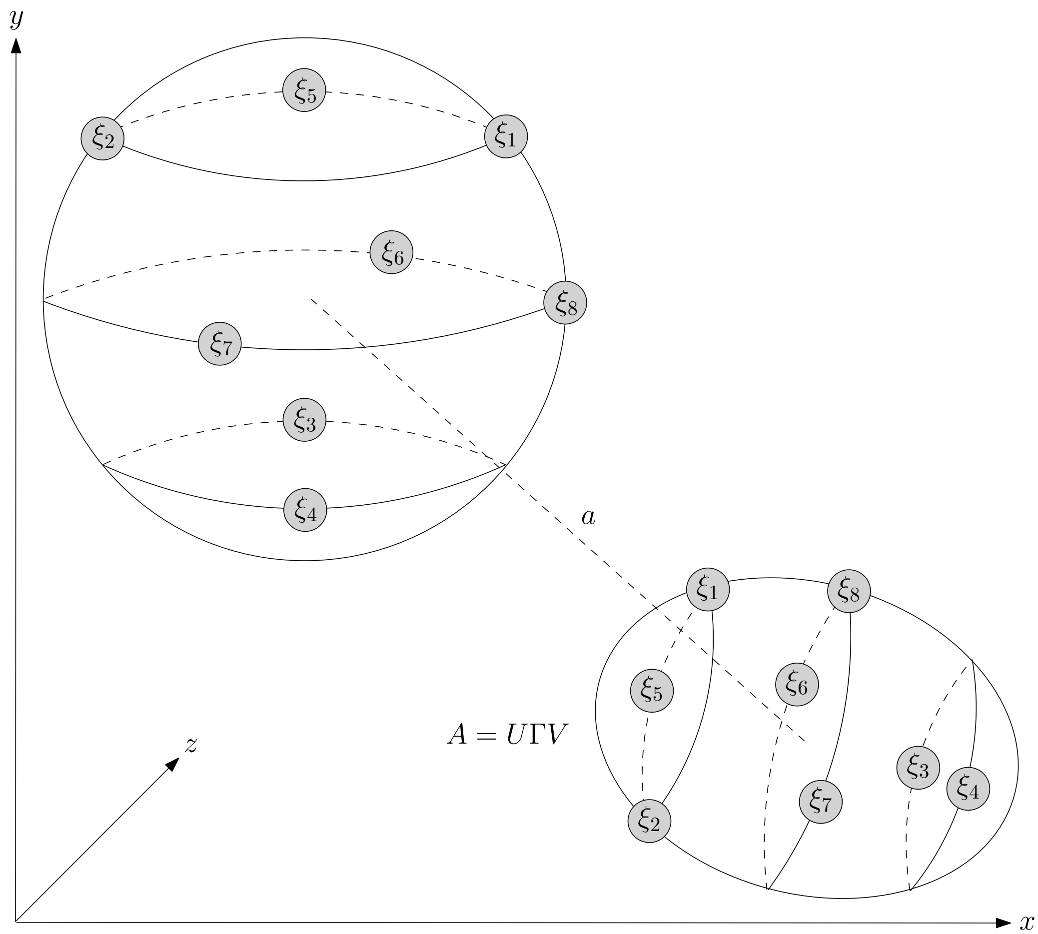

Figure 8.1: Illustration of target configuration and affine configuration

For example, Fig. 8.1 displays a target configuration $\xi = [\xi_1^\top \cdots \xi_8^\top]^\top$ where

$$\xi_1 = \begin{bmatrix} \cos\frac{\pi}{4} \\ 0 \\ \sin\frac{\pi}{4} \end{bmatrix}, \xi_2 = \begin{bmatrix} -\cos\frac{\pi}{4} \\ 0 \\ \sin\frac{\pi}{4} \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ -\cos\frac{\pi}{4} \\ -\sin\frac{\pi}{4} \end{bmatrix}, \xi_4 = \begin{bmatrix} 0 \\ \cos\frac{\pi}{4} \\ -\sin\frac{\pi}{4} \end{bmatrix},$$
$$\xi_5 = \begin{bmatrix} 0 \\ -\cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{bmatrix}, \xi_6 = \begin{bmatrix} \cos\frac{\pi}{3} \\ -\sin\frac{\pi}{3} \\ 0 \end{bmatrix}, \xi_7 = \begin{bmatrix} -\cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} \\ 0 \end{bmatrix}, \xi_8 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

This target configuration consists of eight points on a unit sphere in 3D. Also displayed is another configuration ξ' affine to ξ, as it may be obtained from ξ via rotations and (dimension-wise) scalings via A and a translation via a.

For a given target configuration ξ, let

$$\begin{aligned}\mathcal{A}(\xi) :&= \{\xi' \in \mathbb{R}^{nd} \mid (\exists A \in \mathbb{R}^{d\times d}, \exists a \in \mathbb{R}^d)(\forall i \in [1,n])\xi'_i = A\xi_i + a\} \\ &= \{\xi' \in \mathbb{R}^{nd} \mid (\exists A \in \mathbb{R}^{d\times d}, \exists a \in \mathbb{R}^d)\xi' = (I_n \otimes A)\xi + \mathbf{1}_n \otimes a\} \end{aligned} \tag{8.2}$$

be the family of all configurations affine to ξ. Here $\otimes$ is *Kronecker product*. We say that the n agents with the aggregated state vector $x = [x_1^\top \cdots x_n^\top]^\top \in \mathbb{R}^{nd}$ form an *affine formation* with respect to ξ if $x \in \mathcal{A}(\xi)$.

To achieve an affine formation, consider the distributed control

$$u_i = \sum_{j\in\mathcal{N}_i} w_{ij}(x_j - x_i) \tag{8.3}$$

where the control gain $w_{ij} \in \mathbb{R}$ satisfies

$$\text{(i)}\ \sum_{j\in\mathcal{N}_i} w_{ij}(\xi_j - \xi_i) = 0 \tag{8.4}$$

$$\text{(ii)}\ w_{ij} = \epsilon_i a_{ij}, \quad \epsilon_i \in \mathbb{R}\setminus\{0\}. \tag{8.5}$$

This control (8.3) is in the same form as that for similar formation in Chapter 6: the gains w_{ij} are not simply the edge weights a_{ij}, but are real (nonzero) multiples of a_{ij} (8.5) and satisfy linear constraints with respect to the target configuration ξ (8.4). Different from the control for similar formations where edge weights and control gains are complex, here edge weights and control gains are real.

Moreover, substituting (8.5) into (8.4) and removing the common multiple ϵ_i yield

$$\sum_{j\in\mathcal{N}_i} a_{ij}(\xi_j - \xi_i) = 0. \tag{8.6}$$

This in matrix form is $(L\otimes I_d)\xi = 0$; that is, the target configuration lies in the kernel of $L\otimes I_d$, where L is the signed Laplacian matrix of the (real-)weighted digraph. Since $L\mathbf{1}_n = 0$ (by definition), it follows that

$$\ker(L\otimes I_d) \supseteq \mathcal{A}(\xi). \tag{8.7}$$

To see this, let $\xi' \in \mathcal{A}(\xi)$. Then there exist a matrix A and a vector a such that $\xi' = (I_n\otimes A)\xi + \mathbf{1}_n\otimes a$.

Hence

$$\begin{aligned}(L \otimes I_d)\xi' &= (L \otimes I_d)((I_n \otimes A)\xi + \mathbf{1}_n \otimes a) \\ &= (L \otimes I_d)(I_n \otimes A)\xi + (L \otimes I_d)(\mathbf{1}_n \otimes a) \\ &= (L \otimes A)\xi + (L\mathbf{1}_n) \otimes a \\ &= (I_n \otimes A)(L \otimes I_d)\xi \\ &= 0.\end{aligned}$$

The above derivation means $\xi' \in \ker(L \otimes I_d)$. Therefore, if the control u_i in (8.3) satisfying (8.4) and (8.5) can be found, the kernel of $L \otimes I_d$ at least contains the family of all configurations affine to the target ξ.

Affine Formation Control Problem:

Consider a network of agents modeled by (8.1) interconnected through a digraph, and let $\xi \in \mathbb{R}^{nd}$ be a target configuration (linearly independently from $\mathbf{1}_{nd}$). Design a distributed control u_i in (8.3) such that

(i) $\ker(L \otimes I_d) = \mathcal{A}(\xi)$

(ii) $(\forall x(0) \in \mathbb{R}^{nd})(\exists \xi' \in \mathcal{A}(\xi)) \lim_{t\to\infty} x(t) = \xi'$.

The first requirement (i) strengthens (8.7) to equality; namely the kernel of $L \otimes I_d$ is *exactly* the family $\mathcal{A}(\xi)$ of all configurations affine to ξ. The second requirement (ii) means that every trajectory of the networked agents converges to an affine formation in $\mathcal{A}(\xi)$.

Example 8.1 *We provide an example to illustrate the affine formation control problem. As displayed in Fig. 8.2, eight agents are interconnected through a digraph. The neighbor sets of the agents are* $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}_4 = \emptyset$, $\mathcal{N}_5 = \{1,2,6,7\}$, $\mathcal{N}_6 = \{3,4,7,8\}$, $\mathcal{N}_7 = \{1,5,6,8\}$, *and* $\mathcal{N}_8 = \{4,5,6,7\}$.
Let the target configuration ξ *be eight (three-dimensional) points on a unit sphere (see Fig. 8.1). Thus the family* $\mathcal{A}(\xi)$ *contains all affine formations that can be obtained from* ξ *via affine motions.*
The affine formation control problem is to design a distributed control u_i *in (8.3) such that the kernel of* $L \otimes I_d$ *coincides with* $\mathcal{A}(\xi)$, *and moreover the agents' aggregated state vector asymptotically converges to an affine formation in* $\mathcal{A}(\xi)$.

A necessary graphical condition for solving the affine formation control problem is given below.

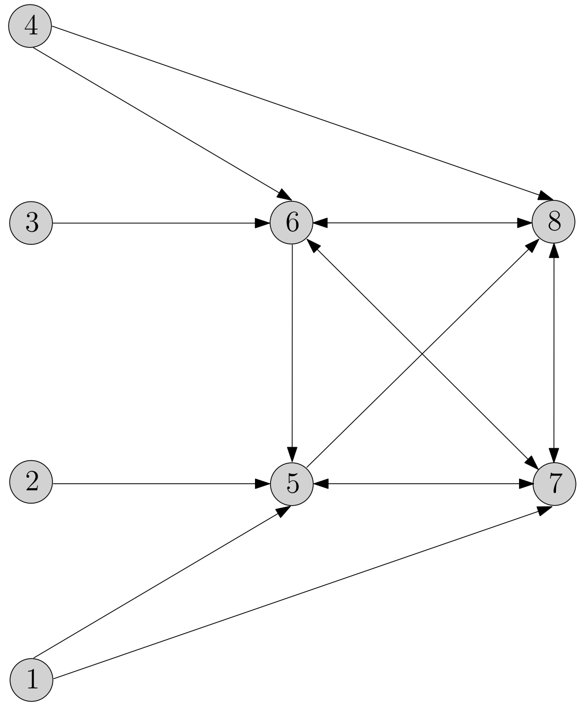

Figure 8.2: Illustrating example of eight agents

Proposition 8.1 *Suppose that there exists a distributed control u_i in (8.3) that solves the affine formation control problem in a d-dimensional space. Then the digraph contains a spanning $(d+1)$-tree.*

Proof. Let $\xi \in \mathbb{R}^{nd}$ be a target configuration. Suppose that there exists a distributed control in (8.3) that solves the d-dimensional affine formation control problem with respect to ξ, but that the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ does *not* contain a spanning $(d+1)$-tree. We will derive a contradiction that $\ker(L \otimes I_d) \supsetneq \mathcal{A}(\xi)$, thereby proving that $\mathcal{G}$ must contain a spanning $(d+1)$-tree.

First, by definition $\mathcal{G}$ containing no spanning $(d+1)$-tree means the following. Let $\mathcal{R}$ be an arbitrary set of $d+1$ nodes. Then removing a set $\mathcal{D}$ of d nodes in $\mathcal{V} \setminus \mathcal{R}$ and all their incoming and outgoing edges, a subset $\mathcal{V}_\mathcal{D} \subsetneq \mathcal{V} \setminus \mathcal{D}$ is unreachable from $\mathcal{R}$ in the new digraph $\mathcal{G}'$. We write this as $\mathcal{R} \not\to \mathcal{V}_\mathcal{D}$ in $\mathcal{G}'$.

Now let $\bar{\mathcal{V}}_\mathcal{D} := \mathcal{V} \setminus (\mathcal{V}_\mathcal{D} \cup \mathcal{D})$. This set $\bar{\mathcal{V}}_\mathcal{D}$ is nonempty because $\mathcal{R} \subseteq \bar{\mathcal{V}}_\mathcal{D}$ (trivially). In addition, even after removing $\mathcal{D}$, the nodes in $\bar{\mathcal{V}}_\mathcal{D}$ can still be reached from $\mathcal{R}$, i.e. $\mathcal{R} \to \bar{\mathcal{V}}_\mathcal{D}$; but $\bar{\mathcal{V}}_\mathcal{D} \not\to \mathcal{V}_\mathcal{D}$.

Let $m := |\mathcal{V}_\mathcal{D}|$ (≥ 1), and relabel

- nodes in $\mathcal{V}_\mathcal{D}$ from v_1 to v_m;

- nodes in $\mathcal{D}$ from v_{m+1} to v_{m+d};
- nodes in $\bar{\mathcal{V}}_{\mathcal{D}}$ from v_{m+d+1} to v_n.

Then the signed Laplacian matrix L of $\mathcal{G}'$ after relabeling (denoted by L') has the following structure:

$$L' = \begin{bmatrix} L'_{11} & L'_{12} & 0 \\ L'_{21} & L'_{22} & L'_{23} \end{bmatrix}.$$

The 0 matrix in the $(1,3)$-block is due to $\bar{\mathcal{V}}_{\mathcal{D}} \not\rightarrow \mathcal{V}_{\mathcal{D}}$ in $\mathcal{G}'$.

Also reorder the components ξ_i of the target formation ξ according to the above relabeling, and denote the result by ξ'. By the assumption that there exists a distributed control in (6.3), we have $(L \otimes I_d)\xi = 0$ and $L\mathbf{1}_n = 0$. Substituting the relabeled L' and ξ' into the two equations yields

$$\left(\begin{bmatrix} L'_{11} & L'_{12} & 0 \end{bmatrix} \otimes I_d\right) \xi' = 0, \quad \begin{bmatrix} L'_{11} & L'_{12} & 0 \end{bmatrix} \mathbf{1}_n = 0.$$

Since ξ' and $\mathbf{1}_{nd}$ are linearly independent (linear independence of ξ and $\mathbf{1}_{nd}$ is assumed in the problem statement), the rows of $[L'_{11}\ L'_{12}\ 0]$ are linearly dependent.

Now remove from L' the $d+1$ rows corresponding to $\mathcal{R}$ and $d+1$ arbitrary columns. Since $\mathcal{R} \subseteq \bar{\mathcal{V}}_{\mathcal{D}}$, it holds that the removed rows have labels in $[m+d+1, n]$. Then the resulting matrix $L'_{\mathcal{R}} \in \mathbb{R}^{(n-d-1)\times(n-d-1)}$ is

$$L'_{\mathcal{R}} = \begin{bmatrix} L'_{\mathcal{R},11} & L'_{\mathcal{R},12} & 0 \\ L'_{\mathcal{R},21} & L'_{\mathcal{R},22} & L'_{\mathcal{R},23} \end{bmatrix}.$$

Thus $[L'_{\mathcal{R},11}\ L'_{\mathcal{R},12}\ 0]$ still has m rows. Since the m rows of $[L'_{11}\ L'_{12}\ 0]$ are linearly dependent, so are the m rows of $[L'_{\mathcal{R},11}\ L'_{\mathcal{R},12}\ 0]$. Hence $L'_{\mathcal{R}}$ has less than $n-d-1$ linearly independent rows, and consequently $\det(L'_{\mathcal{R}}) = 0$.

Finally since the set $\mathcal{R}$ of $d+1$ nodes is arbitrary, the original signed Laplacian matrix L of $\mathcal{G}'$ does not have any minor with size $n-d-1$ that has nonzero determinant. This means that $\operatorname{rank}(L) \leq n-d-2$, and therefore $\ker(L \otimes I_d) \supsetneqq \mathcal{A}(\xi)$. This is a contradiction to the solvability of the affine formation control problem. The proof is now complete. □

Owing to Proposition 8.1, we shall henceforth assume that the digraph contains a spanning $(d+1)$-tree.

Assumption 8.1 *The digraph $\mathcal{G}$ modeling the interconnection structure of the networked agents contains a spanning $(d+1)$-tree.*

Remark 8.1 (Affine formation versus similar formation in 2D) *Consider the special case $d = 2$, i.e. a 2D plane (with two axes labeled x, y). In this case, both affine formations and* *similar formations may be defined, but there is a notable difference. Let $\xi \in \mathbb{C}^n$ or $\mathbb{R}^{2n}$. A similar formation $\xi' \in \mathbb{C}^n$ can be obtained from ξ via a translation, a rotation, and a scaling which is the* same *for both x and y axes. On the other hand, an affine formation $\xi' \in \mathbb{R}^{2n}$ can be obtained from ξ via a translation, a rotation, a scaling for x axis and a possibly* different *scaling for y axis. Hence an affine formation allows different scalings along different axes, and this is the reason why the necessary graphical condition for achieving affine formations requires a spanning 3-tree, in contrast with a spanning 2-tree required for similar formations.*

Even if Assumption 8.1 holds, not every configuration $\xi \in \mathbb{R}^{nd}$ (linearly independent from $\mathbf{1}_{nd}$) whose affine configurations may be achieved by a distributed control u_i in (8.3). An illustrative example is provided below.

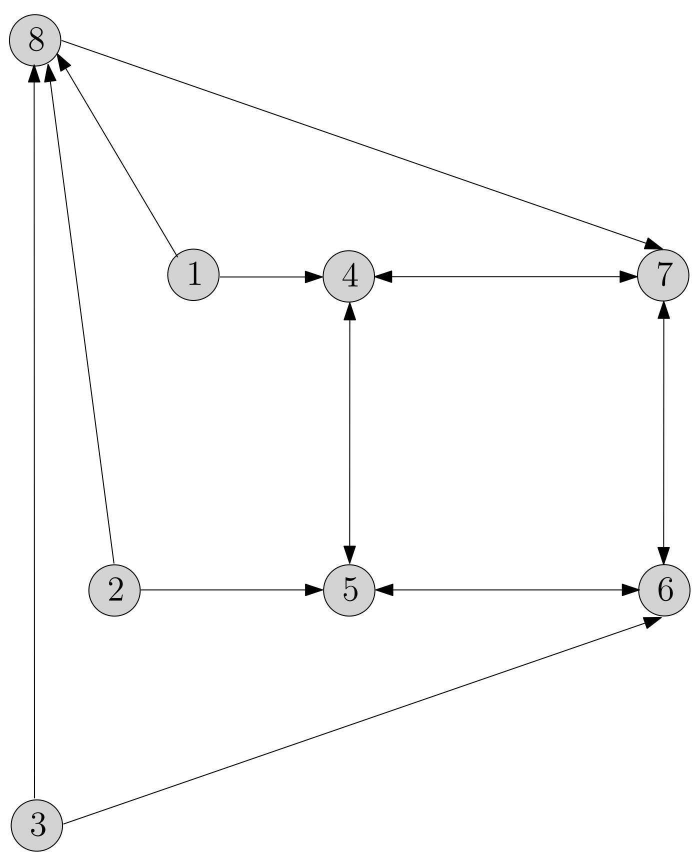

Figure 8.3: Eight-node digraph containing a spanning 3-tree

Example 8.2 *Consider a network of eight agents in a 2D space (i.e. $d = 2$). Their interconnection is modeled by the digraph displayed in Fig. 8.3. This digraph $\mathcal{G}$ contains a spanning 3-tree, with the 3-root subset $\mathcal{R} = \{1, 2, 3\}$. Now consider the following target configuration $\xi = [\xi_1^\top \ \cdots \ \xi_8^\top]^\top$ where*

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \xi_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \xi_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \xi_5 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \xi_6 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \xi_7 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \xi_8 = \begin{bmatrix} 0 \\ -6 \end{bmatrix}.$$

This target configuration ξ has its first seven two-dimensional points on the same line. Thus ξ is not generic, though it is linearly independent from $\mathbf{1}_{16}$. For this non-generic ξ, for every signed Laplacian matrix L of $\mathcal{G}$ with $(L \otimes I_2)\xi = 0$, it is verified that rank$(L) \leq 4$. To see this, write $(L \otimes I_2)\xi$ explicitly as

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{41} & 0 & 0 & l_{44} & l_{45} & 0 & l_{47} & 0 \\ 0 & l_{52} & 0 & l_{54} & l_{55} & l_{56} & 0 & 0 \\ 0 & 0 & l_{63} & 0 & l_{65} & l_{66} & l_{67} & 0 \\ 0 & 0 & 0 & l_{74} & 0 & l_{76} & l_{77} & l_{78} \\ l_{81} & l_{82} & l_{83} & 0 & 0 & 0 & 0 & l_{88} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \end{bmatrix}.$$

For the fourth row of L (other rows are similar), it follows from $L\mathbf{1}_8 = 0$ and $(L \otimes I_2)\xi = 0$ that

$$\begin{aligned} l_{41} + l_{44} + l_{45} + l_{47} &= 0 \\ (l_{41} \otimes I_2)\xi_1 + (l_{44} \otimes I_2)\xi_4 + (l_{45} \otimes I_2)\xi_5 + (l_{47} \otimes I_2)\xi_7 &= 0. \end{aligned}$$

To satisfy these equations, the entries $l_{31}, l_{32}, l_{33}, l_{35}$ are such that

$$\begin{bmatrix} l_{41} \\ l_{44} \\ l_{45} \\ l_{47} \end{bmatrix} \otimes \mathbf{1}_2 = c_4 \begin{bmatrix} \xi_7 - \xi_4 \\ \xi_1 - \xi_5 \\ \xi_4 - \xi_7 \\ \xi_5 - \xi_1 \end{bmatrix} = c_4 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \otimes \mathbf{1}_2$$

for some nonzero real number c_4. Similarly, the (four) entries of rows 5,6,7,8 may be determined up to nonzero real multiples c_5, c_6, c_7, c_8 (respectively). For simplicity, letting

$c_4 = c_5 = c_6 = c_7 = c_8 = 1$ *we have one instance of L as follows:*

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & -1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 & -3 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This L has rank 4, meaning that the last five rows are linearly dependent. Then for arbitrary values of c_4, c_5, c_6, c_7, c_8, these five rows cannot become linearly independent. Hence rank$(L) \leq 4$ for every L with $(L \otimes I_2)\xi = 0$. This means that $\ker(L \otimes I_2) \supsetneqq \mathcal{S}(\xi)$, and consequently there does not exist a distributed control in (8.3) that solves the affine formation control problem with the chosen target configuration ξ.

In virtue of Example 8.2, we henceforth require that the target formation ξ be generic. The requirement is mild, nevertheless, inasmuch as the set of all non-generic configurations has Lebesgue measure zero. This means that for a given non-generic configuration ξ, randomly perturbing its entries generates a generic configuration. It is also noted that every generic configuration ξ is linearly independent from $\mathbf{1}$.

Assumption 8.2 *The target configuration $\xi = [\xi_1^\top \cdots \xi_n^\top]^\top \in \mathbb{R}^{nd}$ is generic.*

8.2 Distributed Algorithm

Example 8.3 *Consider again Example 8.1, where the target configuration ξ consists of eight (three-dimensional) points on a unit sphere (see Fig. 8.1). This ξ is generic.*
To achieve an affine formation of ξ, we consider using the simplest form of the distributed control (8.3) by setting all $\epsilon_i = 1$:

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)), \quad i \in [1, 8] \tag{8.8}$$

where $a_{ij} \in \mathbb{R}$ *are real weights of edges* (j, i) *to be designed to satisfy* (8.6):

$$\sum_{j \in \mathcal{N}_i} a_{ij}(\xi_j - \xi_i) = 0, \quad i \in [1, 8].$$

Now we illustrate how such real weights may be designed. Take agent 6 *for example: it has four neighbors* $3, 4, 7, 8$. *Thus we must find weights* $a_{63}, a_{64}, a_{67}, a_{68}$ *such that*

$$a_{63}(\xi_3 - \xi_6) + a_{64}(\xi_4 - \xi_6) + a_{67}(\xi_7 - \xi_6) + a_{68}(\xi_8 - \xi_6) = 0.$$

Substituting vectors $\xi_3, \xi_4, \xi_6, \xi_7, \xi_8$ *into the above equation yields*

$$a_{63} \begin{bmatrix} -\cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} - \cos\frac{\pi}{4} \\ -\sin\frac{\pi}{4} \end{bmatrix} + a_{64} \begin{bmatrix} -\cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} + \cos\frac{\pi}{4} \\ -\sin\frac{\pi}{4} \end{bmatrix} + a_{67} \begin{bmatrix} -2\cos\frac{\pi}{3} \\ 2\sin\frac{\pi}{3} \\ 0 \end{bmatrix} + a_{68} \begin{bmatrix} 1 - \cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} \\ 0 \end{bmatrix} = 0.$$

This is a system of linear equations, with four unknowns (the weights) and three equations. Thus there are infinitely many solutions (indeed the solution space is one-dimensional). One solution is the following:

$$a_{63} = -\sin\frac{\pi}{3}, \; a_{64} = \sin\frac{\pi}{3}, \; a_{67} = \cos\frac{\pi}{4}(\cos\frac{\pi}{3} - 1), \; a_{68} = -2\cos\frac{\pi}{3}\cos\frac{\pi}{4}.$$

Note that this weight design can be done locally by individual agents if relative information $\xi_j - \xi_i$ $(j \in \mathcal{N}_i)$ *is available.*

Similarly we design other weights to satisfy (8.6), *and write* (8.8) *in vector form:*

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{bmatrix} = \left(\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cos\frac{\pi}{3} - \sin\frac{\pi}{3} & -\cos\frac{\pi}{3} - \sin\frac{\pi}{3} & 0 & 0 & 2\sin\frac{\pi}{3} & -\cos\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\
0 & 0 & -\sin\frac{\pi}{3} & \sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{4}(\cos\frac{\pi}{3} + 1) & \cos\frac{\pi}{4}(\cos\frac{\pi}{3} - 1) & -2\cos\frac{\pi}{3}\cos\frac{\pi}{4} \\
-\sin\frac{\pi}{3} & 0 & 0 & 0 & \sin\frac{\pi}{3} & -\frac{1}{2}\cos\frac{\pi}{4}(1 + \sin\frac{\pi}{3} + \cos\frac{\pi}{3}) & \frac{1}{2}\cos\frac{\pi}{4}(1 - \sin\frac{\pi}{3} - \cos\frac{\pi}{3}) & \cos\frac{\pi}{4}(\sin\frac{\pi}{3} + \cos\frac{\pi}{3}) \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & 0
\end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}.
$$

Inspect that the matrix above has zero row sums, and is indeed the minus of the signed Laplacian matrix L of the (real-)weighted digraph. It is also checked that $(L \otimes I_3)\xi = 0$, namely the target configuration lies in the kernel of $L \otimes I_3$. Moreover, there are exactly four eigenvalues 0 of L, and hence $\ker(L \otimes I_3) = \mathcal{A}(\xi)$ (the first requirement of the affine formation control problem is satisfied).
However, the nonzero eigenvalues of matrix $-L$ are

$$-1.0578, -2.371, 0.3828 + 0.8926\mathrm{j}, 0.3828 - 0.8926\mathrm{j}$$

and hence $-L$ is not stable (the last two eigenvalue have positive real parts). Therefore to *stabilize $x(t)$ to the kernel of $L \otimes I_3$ (to satisfy the second requirement of the affine formation control problem), the unstable eigenvalues of $-L$ must be moved to the open left-half plane. This shows that simply setting all $\epsilon_i = 1$ in (8.3) does not work in general. In fact, ϵ_i need to be properly chosen in order to stabilize $-L$.*

In the following we redescribe the distributed control (8.3) in vector form, and will analyze its stability in relation to the values of ϵ_i in the next section.

Affine Formation Control Algorithm (AFCA):

Every agent i has a state variable $x_i(t) \in \mathbb{R}^d$ $(d \geq 1)$ representing its position in a d-dimensional space at time t; the initial state $x_i(0)$ is an arbitrary d-dimensional real vector. Offline, each agent i computes weights a_{ij} by solving (8.6):

$$\sum_{j \in \mathcal{N}_i} a_{ij}(\xi_j - \xi_i) = 0.$$

Then online, at each time $t \geq 0$, every agent i updates its state $x_i(t)$ using the following distributed control:

$$u_i = \epsilon_i \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i) \tag{8.9}$$

where $\epsilon_i \in \mathbb{R} \setminus \{0\}$ is a (nonzero) real control gain.

Let $x := [x_1^\top \cdots x_n^\top]^\top \in \mathbb{R}^{nd}$ be the aggregated state vector of the networked agents, and $E = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^{n \times n}$ the (diagonal and invertible) control gain matrix. Then the n equations (8.9) become

$$\dot{x} = ((-EL) \otimes I_d)x. \tag{8.10}$$

Remark 8.2 *The above AFCA requires that the following information be available for each individual agent i:*

- *$\xi_j - \xi_i$ for all $j \in \mathcal{N}_i$ (offline computation of weights)*
- *$x_j - x_i$ for all $j \in \mathcal{N}_i$ (online computation of control inputs).*

8.3 Convergence Result

The following is the main result of this section.

Theorem 8.1 *Suppose that Assumptions 8.1 and 8.2 hold. There exists a (diagonal and invertible) control gain matrix* $E = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$ *such that AFCA solves the affine formation control problem.*

To prove Theorem 8.1, we analyze the eigenvalues of the matrix $(-EL) \otimes I_d$ in (8.10). For this, the following fact is useful (which is the real counterpart of Lemma 6.1).

Lemma 8.1 *Consider an arbitrary square real matrix* $M \in \mathbb{R}^{n \times n}$. *If all the principal minors of* M *are nonzero, then there exists an invertible diagonal matrix* $E = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^{n \times n}$ *such that all the eigenvalues of* EM *have positive real parts.*

Proof: The proof is based on induction on n. For the base case $n = 1$, $M = m_{11}$ is a nonzero scalar (as the principal minor of M is nonzero). Let $\epsilon_1 := \frac{1}{m_{11}}$. Then $EM = \epsilon_1 m_{11} = 1 (= \det(E)\det(M))$.

For the induction step, suppose that the conclusion holds for $M \in \mathbb{R}^{(n-1)\times(n-1)}$. Since the $n-1$ eigenvalues are either positive real or conjugate pairs with positive real parts and $\det(E)\det(M) = \lambda_1 \cdots \lambda_{n-1}$, we have $\det(E)\det(M) > 0$. Now consider $M \in \mathbb{R}^{n \times n}$, with all of its principal minors nonzero. Let M_1 be the submatrix of M with the last row and last column removed. Then all the principal minors of M_1 are nonzero, and by the hypothesis there exists an invertible diagonal matrix $E_1 = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_{n-1})$ such that all the eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ of $E_1 M_1$ have positive real parts. Now write

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & m_{nn} \end{bmatrix}$$

where m_{nn} is a nonzero scalar (since all the principal minors of M is nonzero). Also let

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & \epsilon_n \end{bmatrix}$$

for some real ϵ_n. Thus

$$EM = \begin{bmatrix} E_1 & 0 \\ 0 & \epsilon_n \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_3 & m_{nn} \end{bmatrix} = \begin{bmatrix} E_1 M_1 & E_1 M_2 \\ \epsilon_n M_3 & \epsilon_n m_{nn} \end{bmatrix}.$$

If $\epsilon_n = 0$, then

$$EM = \begin{bmatrix} E_1 M_1 & E_1 M_2 \\ 0 & 0 \end{bmatrix}$$