Chapter 6

Similar Formation in Two-Dimensional Space

In this chapter, we introduce a formation control problem of multi-agent systems in two-dimensional (2D) space. The consensus problem studied in Chapter 4 can be viewed as to achieve a special 'point formation', i.e. all the agents reach consensus on their positions in both dimensions respectively. In this sense, the formation control problem in this chapter includes consensus and generalize it to a set of geometric shapes in 2D.

Formation control is an interesting and fundamental topic in teams of autonomous robots, mobile sensors, unmanned aerial vehicles, and autonomous underwater vehicles. Important applications of formation control include source seeking and exploration, map construction, formation flying, and ocean data retrieval. This chapter focuses on formation control in 2D, while 3D formation control will be covered in Chapter 8.

Specifically, the problem studied in this chapter is called *similar formation control*: a network of agents is required to form a geometric shape, which can be obtained from a prescribed desired shape via planar translation, rotation, and scaling. To solve this 2D similar formation control problem, we introduce the second type of graph Laplacian: *complex Laplacian*. Modeling the interacting agents by digraphs, we show that a necessary graphical condition to achieve similar formation is that the digraph contains a *spanning* 2*-tree*, namely there exists (at least) two agents that can reach all the other agents through independent paths. These two root agents play the role of *leaders*, which determine the translation, rotation, and scaling offsets from the prescribed shape. Under this graphical condition, we present a distributed algorithm for the agents to achieve similar formations.

6.1 Problem Statement

Consider a network of n (> 1) agents in a plane (2D space). Each agent i ($\in [1, n]$) has a *state* variable $x_i(t) \in \mathbb{C}$, which is complex and denotes the position of agent i in the plane at time t. Thus $\mathrm{Re}(x_i(\cdot))$ and $\mathrm{Im}(x_i(\cdot))$ are the positions of agents i on the real and imaginary axes, respectively. The time $t \geq 0$ is a (nonnegative) real number and denotes the *continuous* time. The motion of each agent is governed by the following ordinary differential equation:

$$\dot{x}_i = u_i, \quad i \in [1, n] \tag{6.1}$$

where $u_i(t) \in \mathbb{C}$ is the (complex) control input at time t. Thus $\mathrm{Re}(u_i(\cdot))$ (resp. $\mathrm{Im}(u_i(\cdot))$) is the control input along the real axis (resp. imaginary axis).

Let digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ model the interconnection structure of the n agents. Each *node* in $\mathcal{V} = \{1, ..., n\}$ stands for an agent, and each directed *edge* (j, i) in $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes that agent i can measure the *relative position* of agent j (namely $x_j - x_i$ in agent i's coordinate frame). The *neighbor set* of agent i is $\mathcal{N}_i := \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$.

Moreover, consider that digraph $\mathcal{G}$ is weighted: each edge $(j, i) \in \mathcal{V}$ is associated with a complex weight $a_{ij} \in \mathbb{C}$. Hence the adjacency matrix $A = (a_{ij})$, degree matrix $D = \mathrm{diag}(A\mathbf{1})$, and Laplacian matrix $L = D - A$ are all complex.

Define a *target configuration* $\xi = [\xi_1 \cdots \xi_n]^\top \in \mathbb{C}^n$ to be the assignment of the n agents to points in the plane, which specifies the formation *shape* that the agents are required to achieve. Given a target configuration ξ, we say that another configuration ξ' is *similar* to ξ if

$$(\exists \omega_1, \omega_2 \in \mathbb{C})\xi' = \omega_1 \mathbf{1} + \omega_2 \xi.$$

Write $\omega_2 = \rho \mathrm{e}^{\theta}$, $\rho \geq 0$ and $\theta \in [0, 2\pi)$. Then the above means that ξ' is obtained from ξ via (two-dimensional) translation ω_1, rotation θ, and scaling ρ.

For example, Fig. 6.1 displays a target configuration

$$\xi = [1 \;\; \mathrm{e}^{\frac{\pi}{3}\mathrm{j}} \;\; \mathrm{e}^{\frac{2\pi}{3}\mathrm{j}} \;\; \mathrm{e}^{\pi\mathrm{j}} \;\; \mathrm{e}^{\frac{4\pi}{3}\mathrm{j}} \;\; \mathrm{e}^{\frac{5\pi}{3}\mathrm{j}}]^\top$$

which is a regular hexagon. Also displayed is another configuration ξ' similar to ξ, as it can be obtained from ξ via translation ω_1, rotation θ, and scaling ρ.

For a given target configuration ξ, let

$$\mathcal{S}(\xi) := \{\xi' \in \mathbb{C}^n \mid (\exists \omega_1, \omega_2 \in \mathbb{C})\xi' = \omega_1 \mathbf{1} + \omega_2 \xi\} \tag{6.2}$$

be the family of all configurations similar to ξ. Thus $\mathcal{S}(\xi)$ is the (complex) span of the two vectors $\mathbf{1}$ and ξ. If $\xi = c\mathbf{1}$ for some $c \in \mathbb{C}$, then $\mathcal{S}(\xi)$ is degenerated and we are back to consensus in the plane. To consider more general planar formations, we henceforth assume in this chapter that ξ is *linearly independent* from $\mathbf{1}$. Towards the end of this section, we will see that another condition (called 'generic') needs to be imposed on ξ. We say that the n agents with the aggregated state vector $x = [x_1 \cdots x_n]^\top \in \mathbb{C}^n$ form a *similar formation* with respect to ξ if $x \in \mathcal{S}(\xi)$.

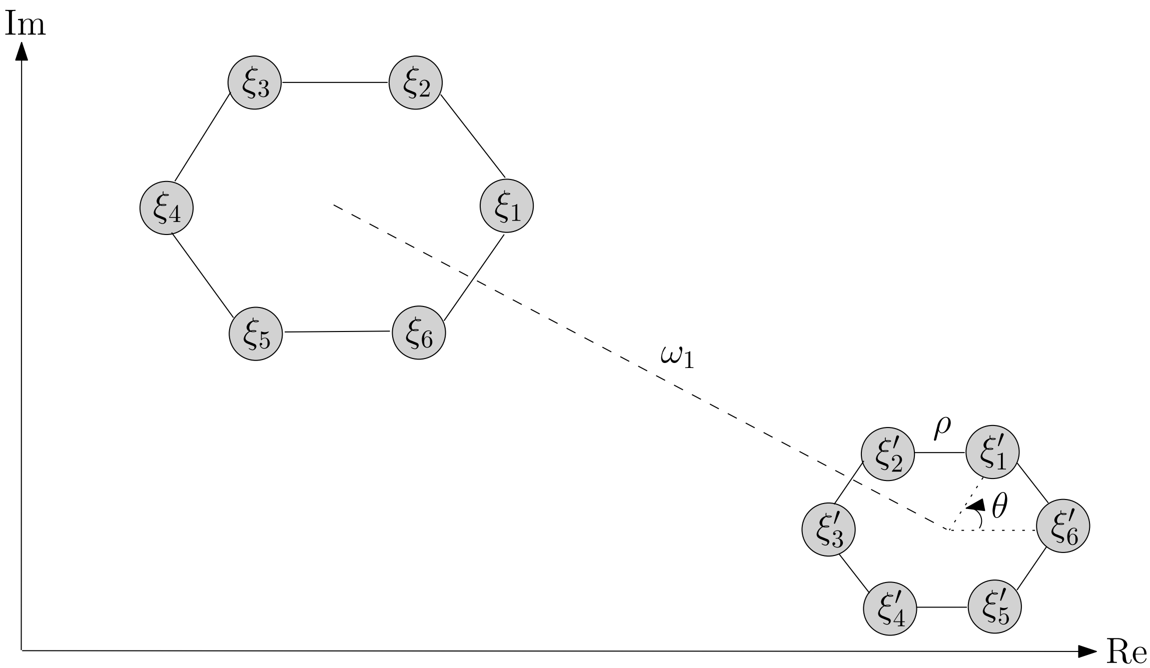

Figure 6.1: Illustration of target configuration and similar configuration

To achieve a similar formation, consider the distributed control

$$u_i = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j - x_i) \tag{6.3}$$

where the control gain $w_{ij} \in \mathbb{C}$ satisfies

$$\text{(i)} \ \sum_{j \in \mathcal{N}_i} w_{ij}(\xi_j - \xi_i) = 0 \tag{6.4}$$

$$\text{(ii)} \ w_{ij} = \epsilon_i a_{ij}, \quad \epsilon_i \in \mathbb{C} \setminus \{0\}. \tag{6.5}$$

This control (6.3) is in the same form as that for consensus, but the gains w_{ij} are not simply the edge weights a_{ij}. Indeed, w_{ij} is a complex (nonzero) multiple of a_{ij} (6.5), and moreover satisfies a linear constraint with respect to the target configuration ξ (6.4).

Substituting (6.5) into (6.4) and removing the common multiple ϵ_i yield

$$\sum_{j \in \mathcal{N}_i} a_{ij}(\xi_j - \xi_i) = 0. \tag{6.6}$$

This in matrix form is $L\xi = 0$; namely the target configuration lies in the kernel of the complex

Laplacian matrix of the (complex-)weighted digraph. Since we also have $L\mathbf{1} = 0$, it follows that

$$\ker L \supseteq \mathcal{S}(\xi). \tag{6.7}$$

Thus if the control in (6.3) satisfying (6.4) and (6.5) can be found, the kernel of the complex Laplacian matrix at least contains the family of all configurations similar to the target ξ.

Similar Formation Control Problem:

Consider a network of agents modeled by (6.1) interconnected through a digraph, and let $\xi \in \mathbb{C}^n$ be a target configuration (linearly independently from $\mathbf{1}$). Design a distributed control u_i in (6.3) such that

$$\text{(i)}\ \ker L = \mathcal{S}(\xi)$$
$$\text{(ii)}\ (\forall x(0) \in \mathbb{C}^n)(\exists \xi' \in \mathcal{S}(\xi)) \lim_{t\to\infty} x(t) = \xi'.$$

The first requirement (i) strengthens (6.7) to equality; namely the kernel of the complex Laplacian matrix is *exactly* the family of all configurations similar to ξ. The second requirement (ii) means that every trajectory of the networked agents converges to a similar formation in $\mathcal{S}(\xi)$.

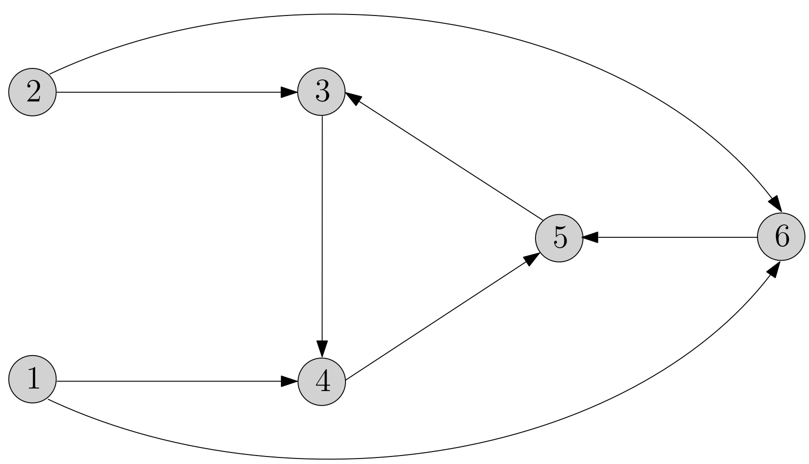

Figure 6.2: Illustrating example of six agents

Example 6.1 *We provide an example to illustrate the similar formation control problem. As displayed in Fig. 6.2, six agents are interconnected through a digraph. The neighbor sets of the agents are* $\mathcal{N}_1 = \mathcal{N}_2 = \emptyset$, $\mathcal{N}_3 = \{2, 5\}$, $\mathcal{N}_4 = \{1, 3\}$, $\mathcal{N}_5 = \{4, 6\}$, *and* $\mathcal{N}_6 = \{1, 2\}$. *Let the target configuration be* $\xi = [1\ \mathrm{e}^{\frac{\pi}{3}\mathrm{j}}\ \mathrm{e}^{\frac{2\pi}{3}\mathrm{j}}\ \mathrm{e}^{\pi\mathrm{j}}\ \mathrm{e}^{\frac{4\pi}{3}\mathrm{j}}\ \mathrm{e}^{\frac{5\pi}{3}\mathrm{j}}]^\top$, *i.e. the desired formation*

shape is a regular hexagon (see Fig. 6.1). Thus the family $\mathcal{S}(\xi)$ contains all hexagons that can be obtained from ξ by translation, rotation, and scaling.
The similar formation control problem is to design a distributed control $u_i(t)$ in (6.3) such that the kernel of the complex Laplacian matrix coincides with $\mathcal{S}(\xi)$, and moreover the agents' aggregated state vector asymptotically converges to a similar formation in $\mathcal{S}(\xi)$.

A necessary graphical condition for solving the similar formation control problem is given below.

Proposition 6.1 *Suppose that there exists a distributed control u_i in (6.3) that solves the similar formation control problem. Then the digraph contains a spanning 2-tree.*

Proof. Let ξ be a target configuration. Suppose that there exists a distributed control in (6.3) that solves the similar formation control problem with respect to ξ, but that the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ does *not* contain a spanning 2-tree. We will derive a contradiction that $\ker L \supsetneqq \mathcal{S}(\xi)$, thereby proving that $\mathcal{G}$ must contain a spanning 2-tree.

First, by definition $\mathcal{G}$ containing no spanning 2-tree means the following. Let $\mathcal{R} = \{v_i, v_j\}$ be a set of arbitrary two nodes. Then after removing a node $v_k \in \mathcal{V}$ and all its incoming and outgoing edges, a subset $\mathcal{V}_k \subsetneqq \mathcal{V} \setminus \{v_k\}$ is unreachable from $\mathcal{R}$ in the new subdigraph $\mathcal{G}'$. We write this as $\mathcal{R} \not\rightarrow \mathcal{V}_k$ in $\mathcal{G}'$.

Now let $\bar{\mathcal{V}}_k := \mathcal{V} \setminus (\mathcal{V}_k \cup \{v_k\})$. This set $\bar{\mathcal{V}}_k$ is nonempty because $\mathcal{R} \subseteq \bar{\mathcal{V}}_k$ (trivially). In addition, even after removing v_k, the nodes in $\bar{\mathcal{V}}_k$ can still be reached from $\mathcal{R}$, i.e. $\mathcal{R} \rightarrow \bar{\mathcal{V}}_k$ in $\mathcal{G}'$; but $\bar{\mathcal{V}}_k \not\rightarrow \mathcal{V}_k$ in $\mathcal{G}'$.

Let $m := |\mathcal{V}_k|$ (≥ 1), and relabel

- nodes $\mathcal{V}_k$ from v_1 to v_m;
- node v_k as v_{m+1};
- nodes in $\bar{\mathcal{V}}_k$ from v_{m+2} to v_n.

Then the complex Laplacian matrix L of $\mathcal{G}'$ after relabeling (denoted by L') has the following structure:

$$L' = \begin{bmatrix} L'_{11} & L'_{12} & 0 \\ L'_{21} & L'_{22} & L'_{23} \end{bmatrix}.$$

The 0 matrix in the $(1,3)$-block is due to $\bar{\mathcal{V}}_k \not\rightarrow \mathcal{V}_k$ in $\mathcal{G}'$.

Also reorder the components of the target configuration ξ according to the above relabeling, and denote the result by

$$\xi' = \begin{bmatrix} \xi'_1 \\ \xi'_2 \\ \xi'_3 \end{bmatrix}.$$

By the assumption that there exists a distributed control in (6.3), we have $L\xi = 0$ and $L\mathbf{1} = 0$. Substituting the relabeled L' and ξ' into the two equations yields

$$\begin{bmatrix} L'_{11} & L'_{12} \end{bmatrix} \begin{bmatrix} \xi'_1 \\ \xi'_2 \end{bmatrix} = 0, \quad \begin{bmatrix} L'_{11} & L'_{12} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = 0.$$

Since ξ' and $\mathbf{1}$ are linearly independent (linear independence of ξ and $\mathbf{1}$ is assumed in the problem statement), so are

$$\begin{bmatrix} \xi'_1 \\ \xi'_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}.$$

Hence the rows of $[L'_{11}\ L'_{12}]$ are linearly dependent.

Now remove from L' the two rows corresponding to $\mathcal{R} = \{v_i, v_j\}$ and two arbitrary columns. We still use indices i, j after the above relabeling, but since $\mathcal{R} \subseteq \bar{\mathcal{V}}_k$, it holds that $i, j \in [m+2, n]$. Then the resulting matrix $L'_{\mathcal{R}} \in \mathbb{C}^{(n-2)\times(n-2)}$ is

$$L'_{\mathcal{R}} = \begin{bmatrix} L'_{\mathcal{R},11} & L'_{\mathcal{R},12} & 0 \\ L'_{\mathcal{R},21} & L'_{\mathcal{R},22} & L'_{\mathcal{R},23} \end{bmatrix}.$$

It follows from $i, j \in [m+2, n]$ that $[L'_{\mathcal{R},11}\ L'_{\mathcal{R},12}]$ have m rows. Since the m rows of $[L'_{11}\ L'_{12}]$ are linearly dependent, so are the m rows of $[L'_{\mathcal{R},11}\ L'_{\mathcal{R},12}]$. Thus $L'_{\mathcal{R}}$ has fewer than $n-2$ linearly independent rows, and $\det(L'_{\mathcal{R}}) = 0$.

Finally since the set $\mathcal{R}$ of two nodes is arbitrary, the original complex Laplacian matrix L of $\mathcal{G}'$ does not have any minor with size $n-2$ that has nonzero determinant. This means that $\text{rank}(L) \le n-3$, and therefore $\ker L \supsetneqq \mathcal{S}(\xi)$. This is a contradiction to the solvability of the similar formation control problem. The proof is now complete. □

Owing to Proposition 6.1, we shall henceforth assume that the digraph contains a spanning 2-tree.

Assumption 6.1 *The digraph $\mathcal{G}$ modeling the interconnection structure of the networked agents contains a spanning* 2*-tree.*

Even if Assumption 6.1 holds, not every configuration ξ (linearly independent from $\mathbf{1}$) whose similar configurations may be achieved by a distributed control u_i in (6.3). The following is such an example.

Example 6.2 *Consider again the six-agent digraph in Fig. 6.2. This digraph $\mathcal{G}$ contains a spanning 2-tree, with the 2-root subset $\mathcal{R} = \{1, 2\}$. Now consider the following target configuration:*

$$\xi = \begin{bmatrix} 0 \\ -3-3\mathrm{j} \\ -1-\mathrm{j} \\ -0.8-1.6\mathrm{j} \\ 1+\mathrm{j} \\ -6\mathrm{j} \end{bmatrix}.$$

While ξ is linearly independent from $\mathbf{1}$, for every complex Laplacian matrix L of $\mathcal{G}$ with $L\xi = 0$, it is verified that rank$(L) \leq 3$. To see this, write $L\xi$ explicitly as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 & l_{35} & 0 \\ l_{41} & 0 & l_{43} & l_{44} & 0 & 0 \\ 0 & 0 & 0 & l_{54} & l_{55} & l_{56} \\ l_{61} & l_{62} & 0 & 0 & 0 & l_{66} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix}.$$

For the third row (other rows are similar), it follows from $L\mathbf{1} = 0$ and $L\xi = 0$ that

$$l_{32} + l_{33} + l_{35} = 0$$
$$l_{32}\xi_2 + l_{33}\xi_3 + l_{35}\xi_5 = 0.$$

To satisfy these two equations, the entries l_{32}, l_{33}, l_{35} are such that

$$\begin{bmatrix} l_{32} \\ l_{33} \\ l_{35} \end{bmatrix} = c_3 \begin{bmatrix} \xi_5 - \xi_3 \\ \xi_2 - \xi_5 \\ \xi_3 - \xi_2 \end{bmatrix} = c_3 \begin{bmatrix} 2+2\mathrm{j} \\ -4-4\mathrm{j} \\ 2+2\mathrm{j} \end{bmatrix}$$

for some nonzero complex number c_3. Similarly, the (three) entries of rows 4,5,6 may be determined up to nonzero complex multiples c_4, c_5, c_6 (respectively). For simplicity, letting

$c_3 = c_4 = c_5 = c_6 = 1$ *we have one instance of* L *as follows:*

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+2\mathrm{j} & -4-4\mathrm{j} & 0 & 2+2\mathrm{j} & 0 \\ 0.2-0.6\mathrm{j} & 0 & 0.8+1.6\mathrm{j} & -1-\mathrm{j} & 0 & 0 \\ 0 & 0 & 0 & -1-7\mathrm{j} & -0.8+4.4\mathrm{j} & 1.8+2.6\mathrm{j} \\ 3-3\mathrm{j} & 6\mathrm{j} & 0 & 0 & 0 & -3-3\mathrm{j} \end{bmatrix}.$$

This L *has rank* 3*, meaning that the last four rows are linearly dependent. Then for arbitrary values of* c_3, c_4, c_5, c_6*, these four rows cannot become linearly independent. Hence* rank$(L) \leq 3$ *for every* L *with* $L\xi = 0$*. This means that* $\ker L \supsetneqq \mathcal{S}(\xi)$*, and consequently there does not exist a distributed control in (6.3) that solves the similar formation control problem with the chosen target configuration* ξ*.*

The target configuration ξ in the above example satisfies a linear algebraic equation with integer coefficients:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -3-3\mathrm{j} \\ -1-\mathrm{j} \\ -0.8-1.6\mathrm{j} \\ 1+\mathrm{j} \\ -6\mathrm{j} \end{bmatrix} = 0.$$

Such a configuration ξ is called *non-generic*. Geometrically, in the plane there are four components of ξ (1st, 2nd, 3rd, and 5th) on the same line.

Since Example 6.2 shows a case where similar formations of a non-generic configuration cannot be achievable on a digraph containing a spanning 2-tree, we henceforth require that the target configuration be generic. A configuration $\xi = [\xi_1 \cdots \xi_n]^\top \in \mathbb{C}^n$ is said to be *generic* if ξ_i's do not satisfy any nontrivial algebraic equation with integer coefficients. Intuitively speaking, a generic configuration has no degeneracy: in 2D, no three points on the same line and no three lines through the same point. As a consequence, any generic configuration ξ is linearly independent from $\mathbf{1}$.

It is noted, however, that not all non-generic configurations whose similar configurations cannot be achieved. In fact, if the digraph considered in Example 6.2 had one more edge $(1, 3)$, the non-generic configuration ξ's similar configurations could be achievable. Indeed, following the same procedure described in Example 6.2, with a new edge $(1, 3)$ we derive an instance of the new

Laplacian matrix below:

$$L' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2+2\mathrm{j} & -4-4\mathrm{j} & 0 & 2+2\mathrm{j} & 0 \\ 0.2-0.6\mathrm{j} & 0 & 0.8+1.6\mathrm{j} & -1-\mathrm{j} & 0 & 0 \\ 0 & 0 & 0 & -1-7\mathrm{j} & -0.8+4.4\mathrm{j} & 1.8+2.6\mathrm{j} \\ 3-3\mathrm{j} & 6\mathrm{j} & 0 & 0 & 0 & -3-3\mathrm{j} \end{bmatrix}.$$

The only change is the $(3,1)$-entry from 0 to 1, owing to the added edge $(1,3)$. This L' has rank 4; therefore $\ker L' = \mathcal{S}(\xi)$. Thus one may consider imposing further digraph connectivity to deal with non-generic configurations.

On the other hand, the set of all non-generic configurations has Lebesgue measure zero, because random perturbations destroy integer-coefficient algebraic equations. This means that for a given non-generic configuration ξ (e.g. the one in Example 6.2), randomly perturbing its components generates a generic configuration. For this reason, we assume that the target configuration ξ is generic.

Assumption 6.2 *The target configuration $\xi = [\xi_1 \cdots \xi_n]^\top \in \mathbb{C}^n$ is generic.*

Remark 6.1 (Global versus local coordinate frames) *We end this section with a discussion on the local coordinate frames of the agents with respect to the global coordinate frame. So far the state x_i and control u_i of agent i that we have discussed are in the global coordinate frame Σ. In formation control, the agents are usually robots with onboard sensors, thus having their own local coordinate frames that are not necessarily aligned with the global Σ and time-varying. For distributed control, knowledge of Σ is often not available and thus should not be assumed. Let the local frame of agent i at time t be $\Sigma_i(t)$, whose orientation is $\theta_i(t)$ counterclockwise from the orientation of Σ. Also let $x_{i,\mathrm{loc}}(t)$ and $u_{i,\mathrm{loc}}(t)$ be (respectively) the state and control at time t of agent i in $\Sigma_i(t)$. Then*

$$x_i(t) = x_{i,\mathrm{loc}}(t)e^{-j\theta_i(t)}$$
$$u_i(t) = u_{i,\mathrm{loc}}(t)e^{-j\theta_i(t)}.$$

Recall from (6.3) that

$$u_i(t) = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j(t) - x_i(t)).$$

Substituting the above equation of $x_i(t)$ into the right-hand side yields

$$u_i(t) = \sum_{j\in\mathcal{N}_i} w_{ij}(x_{j,\mathrm{loc}}(t)e^{-j\theta_i(t)} - x_{i,\mathrm{loc}}(t)e^{-j\theta_i(t)})$$
$$= \sum_{j\in\mathcal{N}_i} w_{ij}(x_{j,\mathrm{loc}}(t) - x_{i,\mathrm{loc}}(t))e^{-j\theta_i(t)}.$$

Now equating the right-hand sides of the above two $u_i(t)$-equations, we derive

$$u_{i,\mathrm{loc}}(t) = \sum_{j\in\mathcal{N}_i} w_{ij}(x_{j,\mathrm{loc}}(t) - x_{i,\mathrm{loc}}(t)).$$

This shows that the control $u_{i,\mathrm{loc}}(t)$ in the local $\Sigma_i(t)$ is unaffected by the time-varying orientation difference from the global Σ. Hence the control u_i in (6.3), though with respect to the global frame Σ, may be implemented in agent i's local frame $\Sigma_i(t)$ (as $u_{i,\mathrm{loc}}$) based on the state difference $x_{j,\mathrm{loc}}-x_{i,\mathrm{loc}}$ in $\Sigma_i(t)$ as well. With this justification and for simplicity, we will write u_i, x_i (instead of $u_{i,\mathrm{loc}}$, $x_{i,\mathrm{loc}}$).

6.2 Distributed Algorithm

Example 6.3 *Consider again Example 6.1, where the target configuration is the regular hexagon $\xi = [1\ \mathrm{e}^{\frac{\pi}{3}\mathrm{j}}\ \mathrm{e}^{\frac{2\pi}{3}\mathrm{j}}\ \mathrm{e}^{\pi\mathrm{j}}\ \mathrm{e}^{\frac{4\pi}{3}\mathrm{j}}\ \mathrm{e}^{\frac{5\pi}{3}\mathrm{j}}]^\top$. This ξ is generic.*
To achieve a similar formation of ξ, we consider using the simplest form of the distributed control (6.3) by setting all $\epsilon_i = 1$:

$$\dot{x}_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)), \quad i \in [1, 6] \tag{6.8}$$

where $a_{ij} \in \mathbb{C}$ are complex weights of edges (j, i) to be designed to satisfy (6.6):

$$\sum_{j\in\mathcal{N}_i} a_{ij}(\xi_j - \xi_i) = 0, \quad i \in [1, 6].$$

In Fig. 6.3, we illustrate how such complex weights may be designed. For agent 3, it has two neighbors 2, 5. Thus we need to find weights a_{32}, a_{52} such that

$$a_{32}(\xi_2 - \xi_3) + a_{35}(\xi_5 - \xi_3) = 0.$$

Writing a_{32}, a_{52} in polar coordinates, the above equation may be satisfied through making *proper rotations and scalings (dashed arrows in Fig. 6.3), i.e.*

$$\rho_{32}e^{\theta_{32}j}(\xi_2 - \xi_3) + \rho_{35}e^{\theta_{35}j}(\xi_5 - \xi_3) = 0.$$

There are infinitely many choices; a simple one is $\rho_{32} = \sqrt{3}, \theta_{32} = 0$ *and* $\rho_{35} = 1, \theta_{35} = -\frac{\pi}{2}$. *Hence* $w_{32} = \sqrt{3}$, $w_{35} = -j$. *Note that this weight design can be done locally by individual agents if relative information* $\xi_j - \xi_i$ $(j \in \mathcal{N}_i)$ *is available.*
Similarly we design other complex weights to satisfy (6.6), and write (6.8) in vector form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & -\sqrt{3}+j & 0 & -j & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2}+\frac{\sqrt{3}}{2}j & -\frac{\sqrt{3}}{2}j & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}+\frac{\sqrt{3}}{2}j & -\frac{3}{2}-\frac{\sqrt{3}}{2}j & 1 \\ -\frac{3}{2}-\frac{\sqrt{3}}{2}j & 1 & 0 & 0 & 0 & \frac{1}{2}+\frac{\sqrt{3}}{2}j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.$$

Inspect that the matrix above has zero row sums, and is indeed the minus of the complex Laplacian matrix L *of the (complex) weighted digraph. It is also checked that* $L\xi = 0$, *namely the target configuration lies in the kernel of* L. *Moreover, there are exactly two eigenvalues* 0 *of* L, *and hence* $\ker L = \mathcal{S}(\xi)$ *(the first requirement of the similar formation control problem is satisfied).*
However, the nonzero eigenvalues of matrix $-L$ *are*

$$-1.917 + 0.8963j, -1.1283 - 1.042j, -0.1867 - 0.5863j, 0.5 + 0.866j$$

and hence $-L$ *is not stable (the last eigenvalue has positive real part). Therefore to stabilize* $x(t)$ *to the kernel of* L *(to satisfy the second requirement of the similar formation control problem), the unstable eigenvalues of* $-L$ *must be moved to the open left-half plane. This shows that simply setting all* $\epsilon_i = 1$ *in (6.3) does not work in general. In fact,* ϵ_i *need to be properly chosen in order to stabilize* $-L$.

In the following we redescribe the distributed control (6.3) in vector form, and will analyze its stability in relation to the values of ϵ_i in the next section.

Similar Formation Control Algorithm (SFCA):

Every agent i has a state variable $x_i(t) \in \mathbb{C}$ representing its position in 2D at time $t \geq 0$; the initial state $x_i(0)$ is an arbitrary complex number. Offline, each agent i computes weights

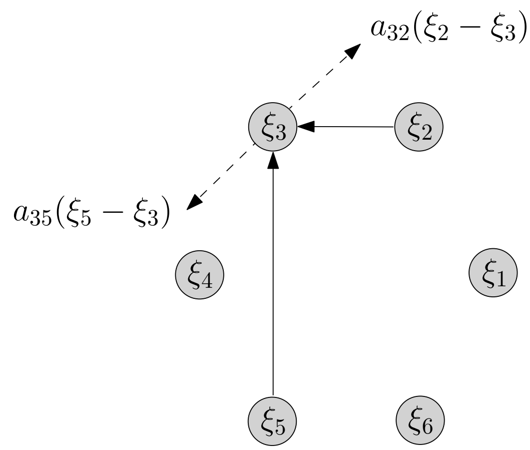

Figure 6.3: Illustration of design of complex weights

$a_{ij} = \rho_{ij} \mathrm{e}^{\theta_{ij}}$ by solving

$$\sum_{j \in \mathcal{N}_i} \rho_{ij} \mathrm{e}^{\theta_{ij}} (\xi_j - \xi_i) = 0 \tag{6.9}$$

such that (6.6) holds. Then online, at each time $t \geq 0$, every agent i updates its state $x_i(t)$ using the following distributed control:

$$u_i = \epsilon_i \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i) \tag{6.10}$$

where $\epsilon_i \in \mathbb{C} \setminus \{0\}$ is a (nonzero) complex control gain.

Let $x := [x_1 \cdots x_n]^\top \in \mathbb{C}^n$ be the aggregated state vector of the networked agents, and $E = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n) \in \mathbb{C}^{n \times n}$ the (diagonal and invertible) control gain matrix. Then the n equations (6.10) become

$$\dot{x} = (-EL)x. \tag{6.11}$$

Remark 6.2 *The above SFCA requires that the following information be available for each individual agent i:*

- $\xi_j - \xi_i$ *for all* $j \in \mathcal{N}_i$ *(offline computation of weights)*
- $x_j - x_i$ *for all* $j \in \mathcal{N}_i$ *(online computation of control inputs).*

6.3 Convergence Result

The following is the main result of this section.

Theorem 6.1 *Suppose that Assumptions 6.1 and 6.2 hold. There exists a (diagonal and invertible) control gain matrix* $E = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$ *such that SFCA solves the similar formation control problem.*

To prove Theorem 6.1, we analyze the eigenvalues of the matrix $-EL$ in (6.11). For this, the following fact is useful.

Lemma 6.1 *Consider an arbitrary square complex matrix* $M \in \mathbb{C}^{n\times n}$*. If all the principal minors of* M *are nonzero, then there exists an invertible diagonal matrix* $E = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n) \in \mathbb{C}^{n\times n}$ *such that all the eigenvalues of* EM *have positive real parts.*

Proof: The proof is based on induction on n. For the base case $n = 1$, $M = m_{11}$ is a nonzero scalar (as the principal minor of M is nonzero). Write $m_{11} = \rho_1 \mathrm{e}^{\mathrm{j}\theta_1}$, and let $\epsilon_1 := \gamma_1 \mathrm{e}^{\mathrm{j}\phi_1}$ where $\gamma_1 \neq 0$ and ϕ_1 is such that $(\phi_1 + \theta_1)(\mathrm{mod}\ 2\pi) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $EM = \epsilon_1 m_{11} = \rho_1\gamma_1 \mathrm{e}^{\mathrm{j}(\phi_1+\theta_1)}$, which has positive real part.

For the induction step, suppose that the conclusion holds for $M \in \mathbb{C}^{(n-1)\times(n-1)}$. Now consider $M \in \mathbb{C}^{n\times n}$, with all of its principal minors nonzero. Let M_1 be the submatrix of M with the last row and last column removed. Then all the principal minors of M_1 are nonzero, and by the hypothesis there exists an invertible diagonal matrix $E_1 = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_{n-1})$ such that all the eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ of $E_1 M_1$ have positive real parts. Now write

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & m_{nn} \end{bmatrix}$$

where m_{nn} is a nonzero scalar (since all the principal minors of M are nonzero). Also let

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & \epsilon_n \end{bmatrix}$$

for some complex ϵ_n. Thus

$$EM = \begin{bmatrix} E_1 & 0 \\ 0 & \epsilon_n \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_3 & m_{nn} \end{bmatrix} = \begin{bmatrix} E_1 M_1 & E_1 M_2 \\ \epsilon_n M_3 & \epsilon_n m_{nn} \end{bmatrix}.$$